# Finite Automata

Part Two

# Recap from Last Time

# DFAs

- A ***DFA*** is a
  - ***D***eterministic
  - ***F***inite
  - ***A***utomaton
- DFAs are the simplest type of automaton that we will see in this course.

# DFAs

- A DFA is defined relative to some alphabet Σ.
- For each state in the DFA, there must be ***exactly one*** transition defined for each symbol in Σ.
  - This is the "deterministic" part of DFA.
- There is a unique start state.
- There are zero or more accepting states.

If $D$ is a DFA, the ***language of D***, denoted $\mathcal{L}(D)$, is $\{ w \in \Sigma^* \mid D \text{ accepts } w \}$.

A language $L$ is called a ***regular language*** if there exists a DFA $D$ such that $\mathcal{L}(D) = L$.

# NFAs

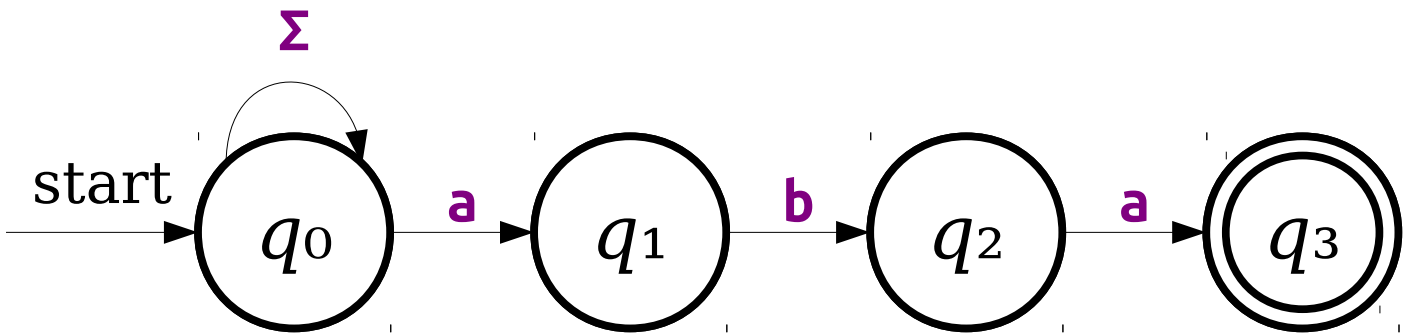
- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- Can have missing transitions or multiple transitions defined on the same input symbol.
- Accepts if *any possible series of choices* leads to an accepting state.

New Stuff!

# Intuiting Nondeterminism

- Nondeterministic machines are a serious departure from physical computers. How can we build up an intuition for them?
- There are two particularly useful frameworks for interpreting nondeterminism:
  - ***Perfect positive guessing***
  - ***Massive parallelism***

# Perfect Positive Guessing



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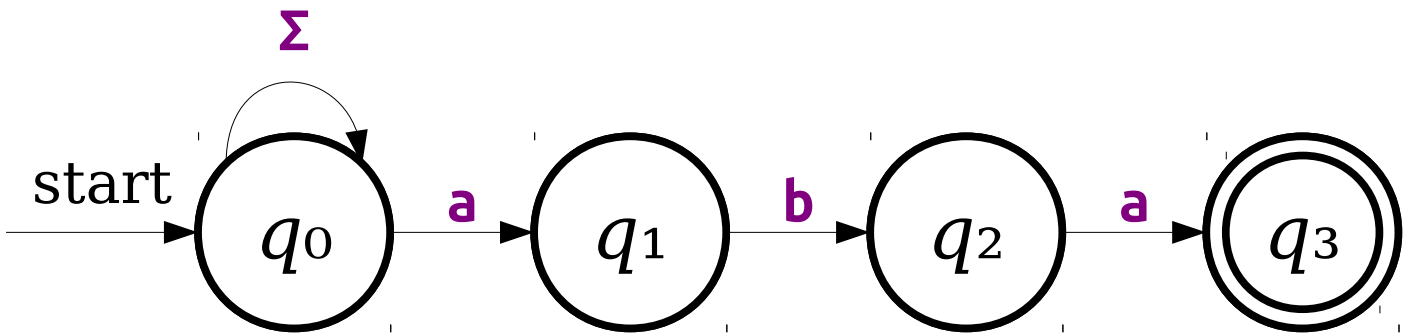


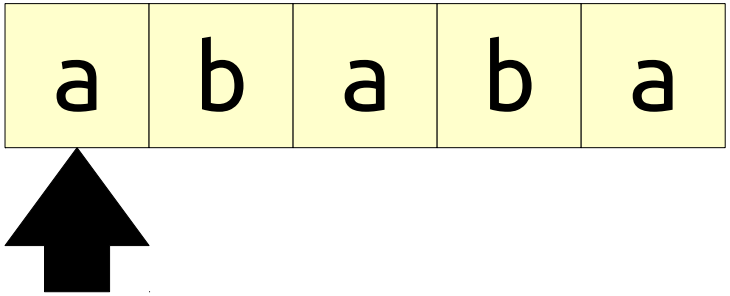

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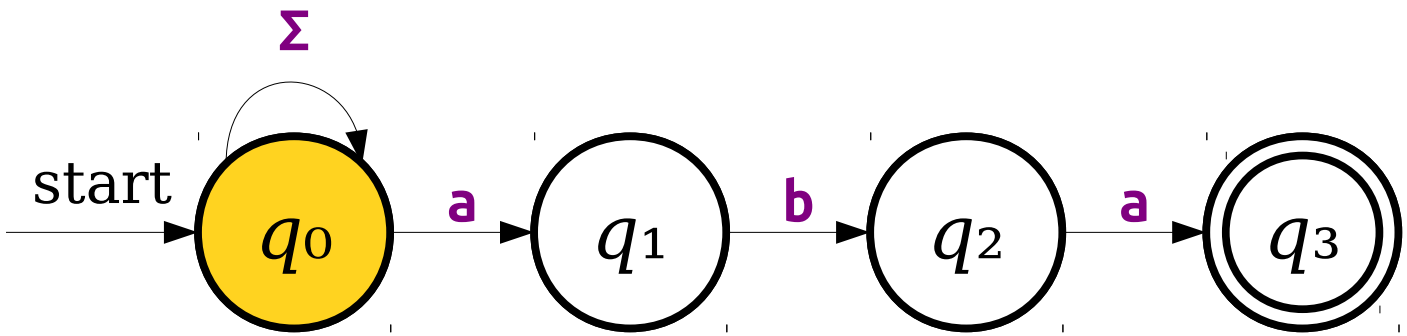


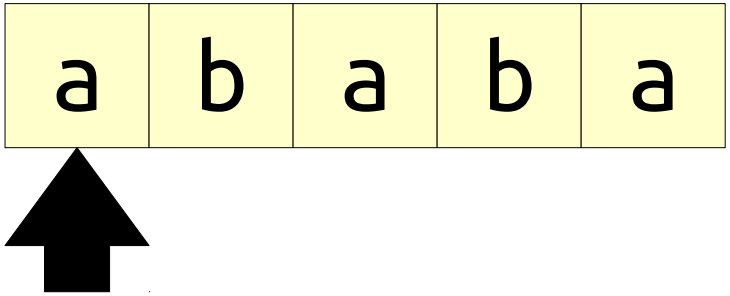

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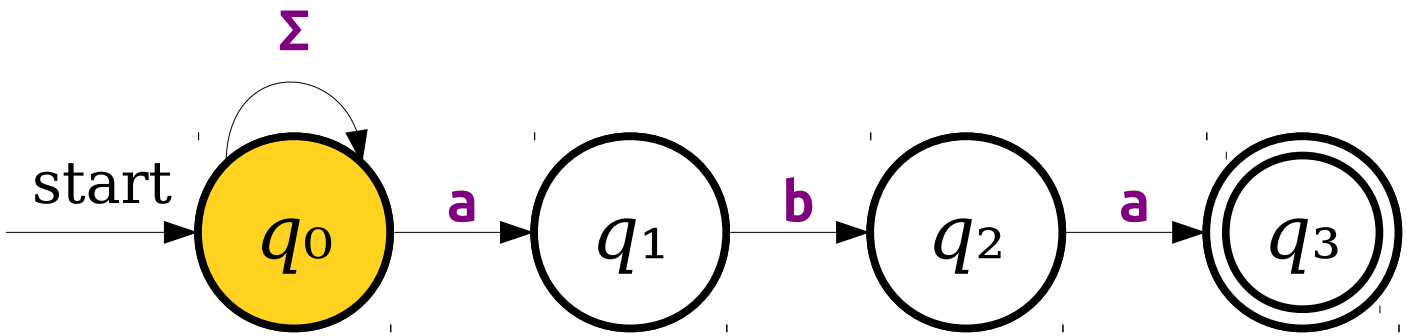


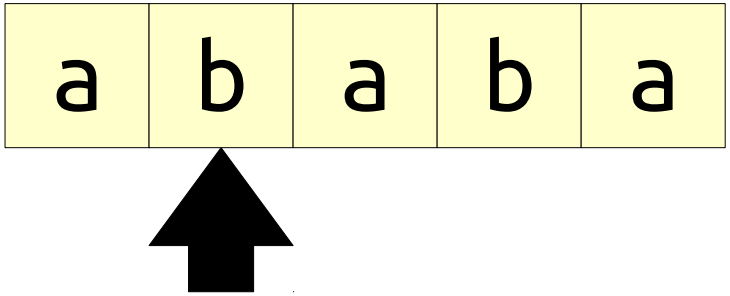

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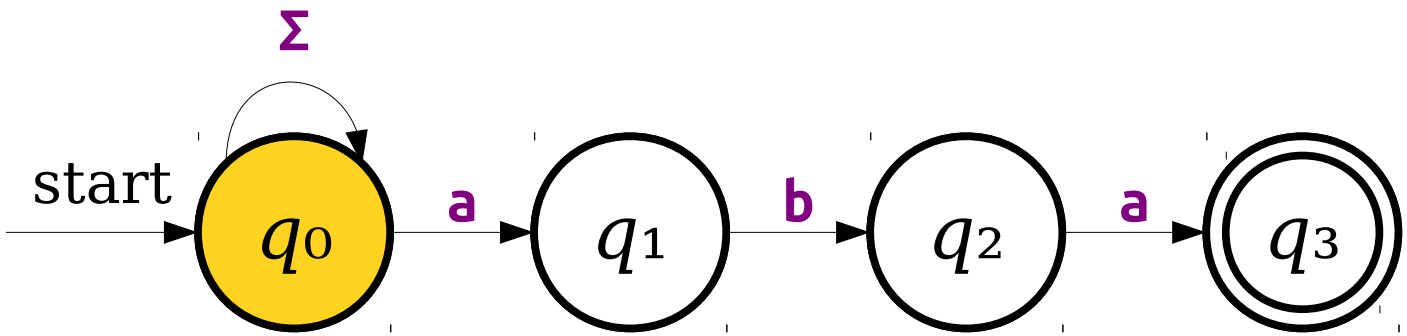


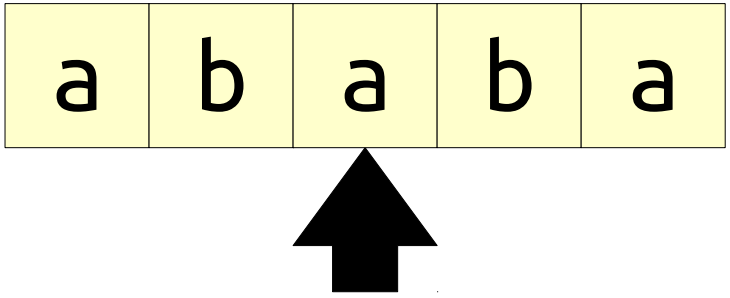

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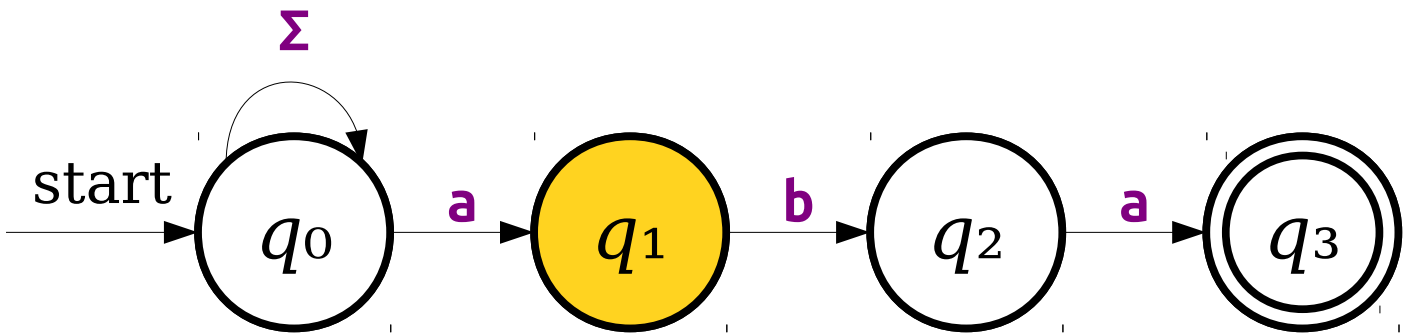


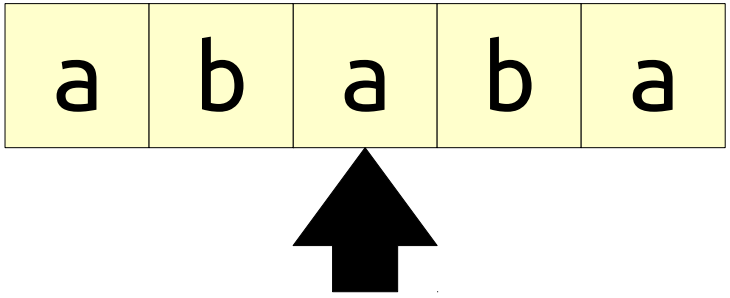

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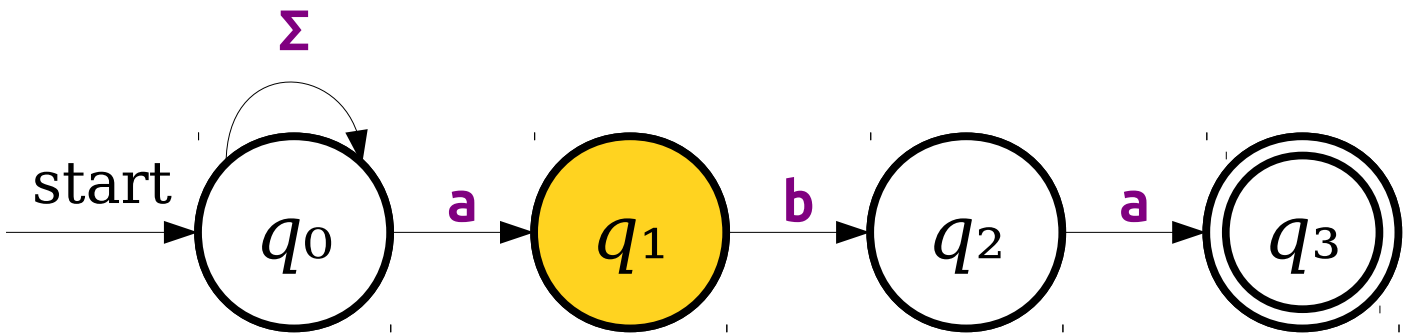


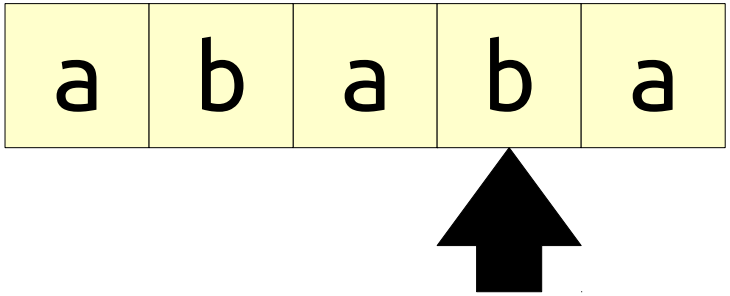

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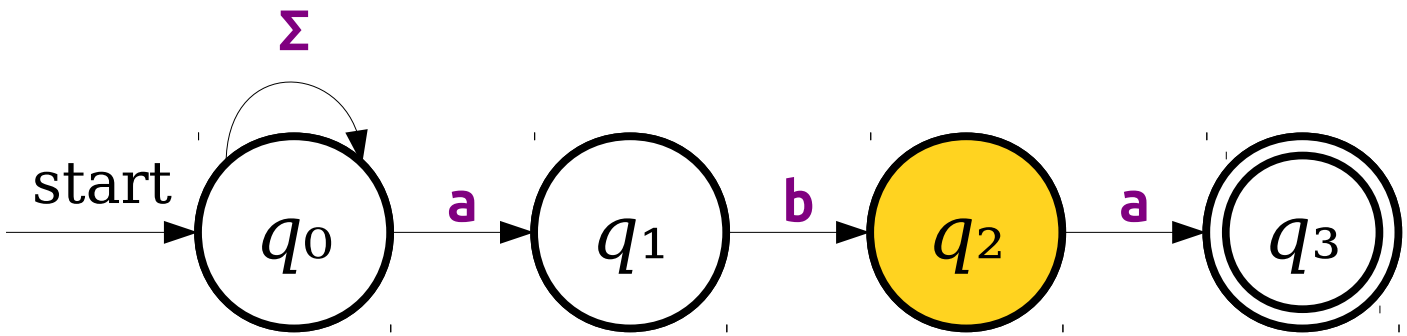


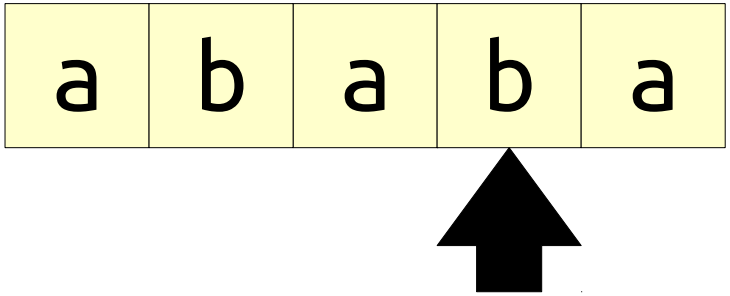

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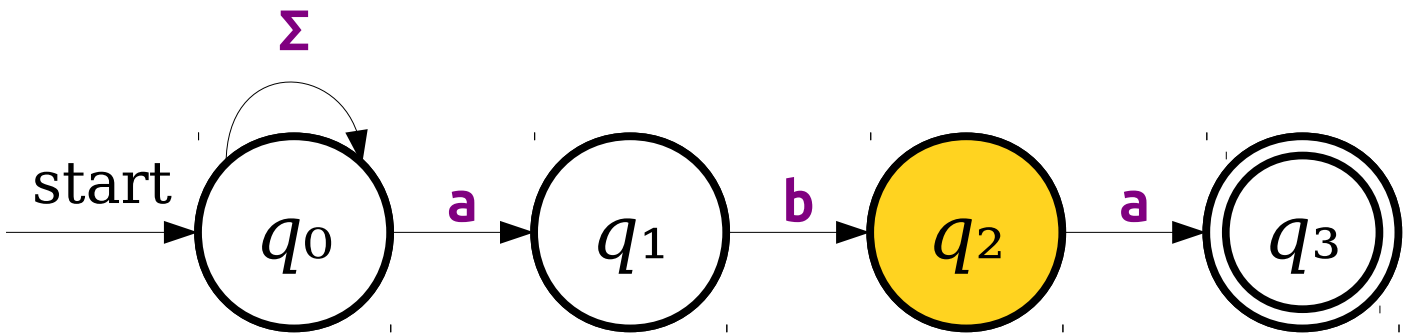


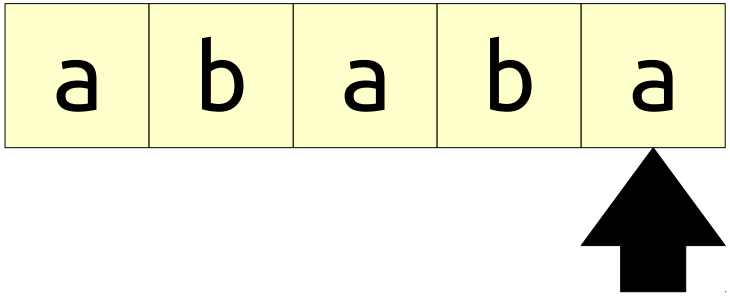

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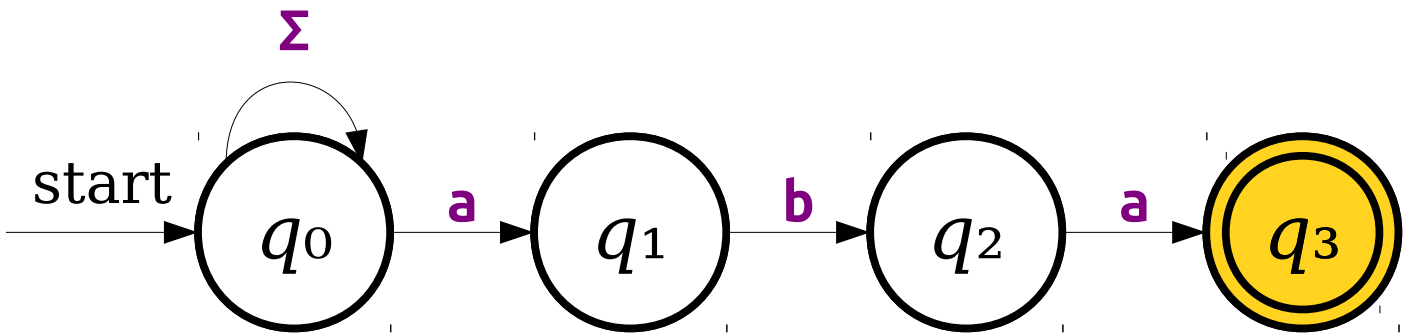


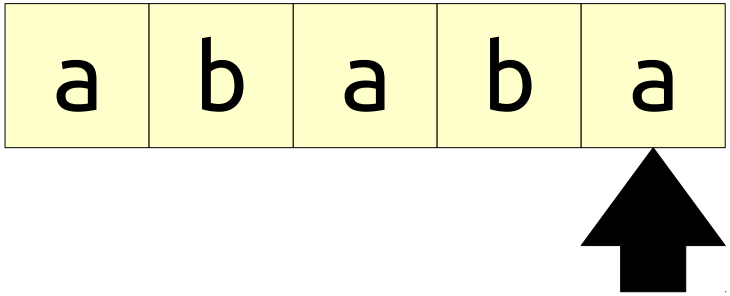

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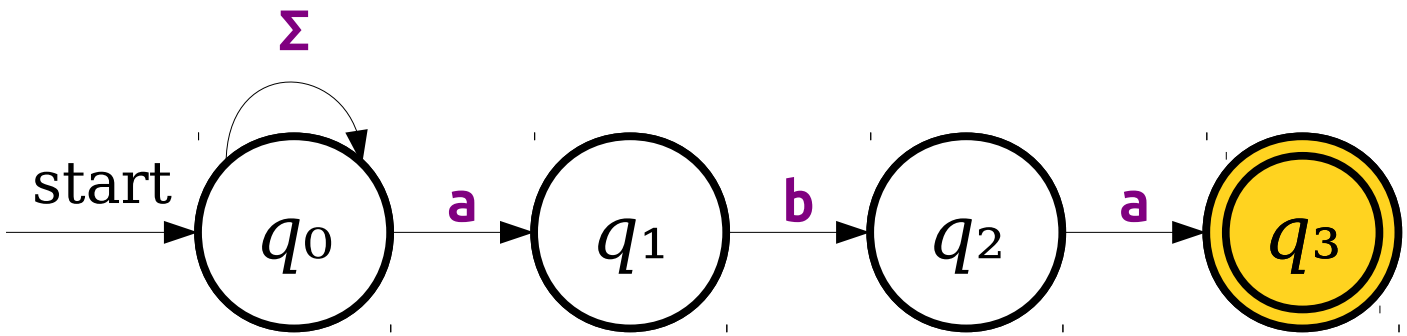


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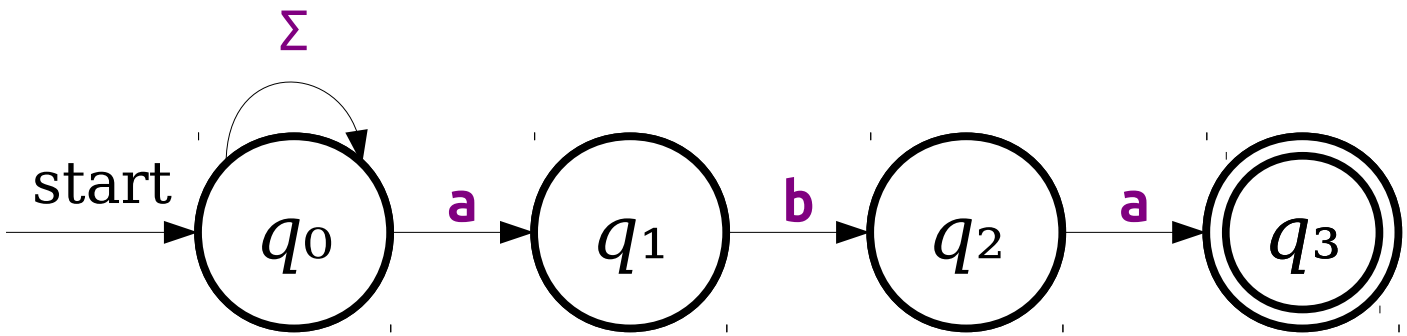
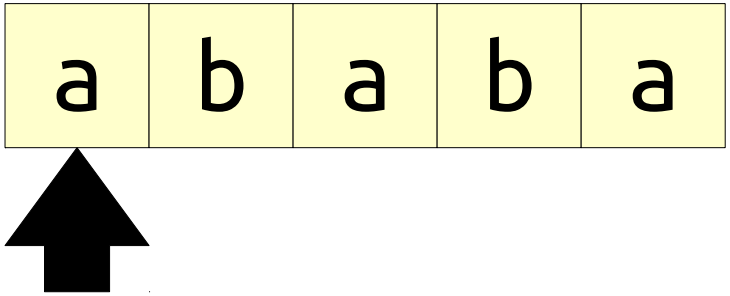
# Perfect Positive Guessing

- We can view nondeterministic machines as having ***Magic Superpowers*** that enable them to guess choices that lead to an accepting state.
  - If there is at least one choice that leads to an accepting state, the machine will guess it.
  - If there are no choices, the machine guesses any one of the wrong guesses.
- There is no known way to physically model this intuition of nondeterminism – this is quite a departure from reality!

# Massive Parallelism



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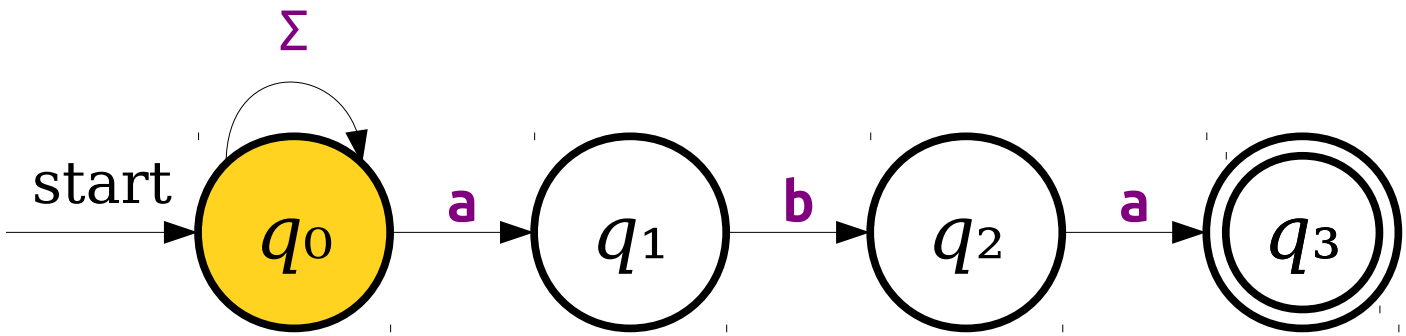


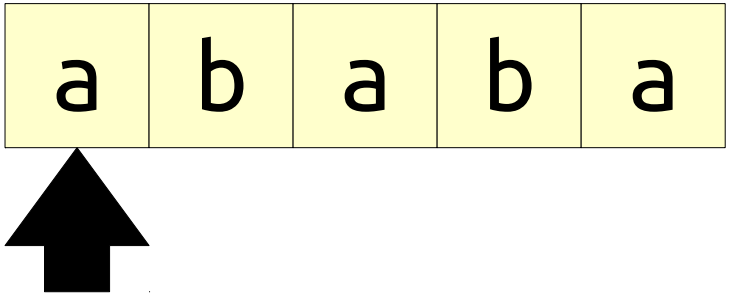

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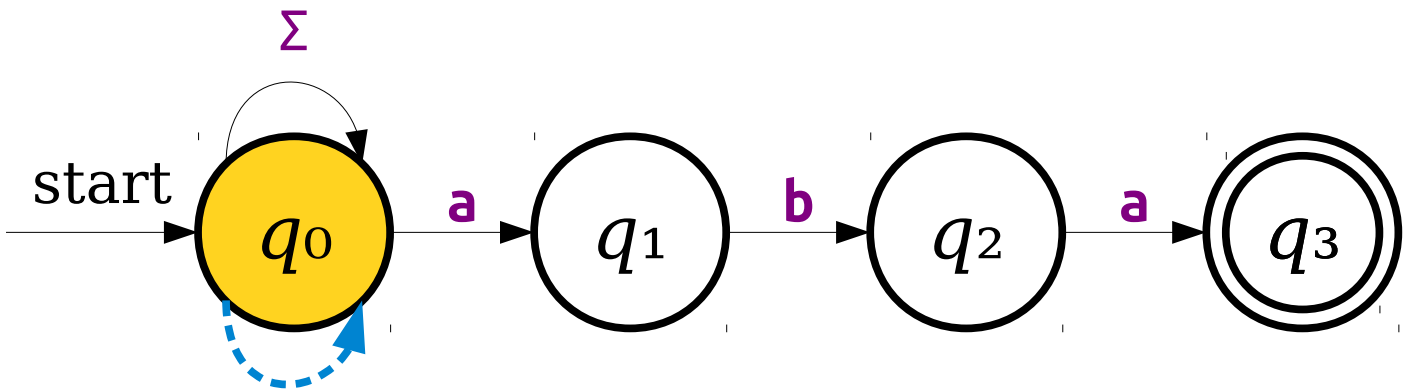


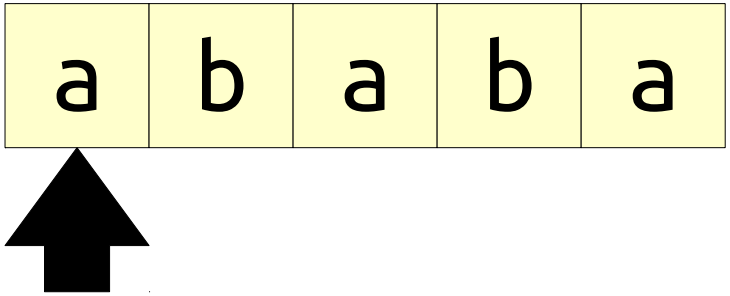

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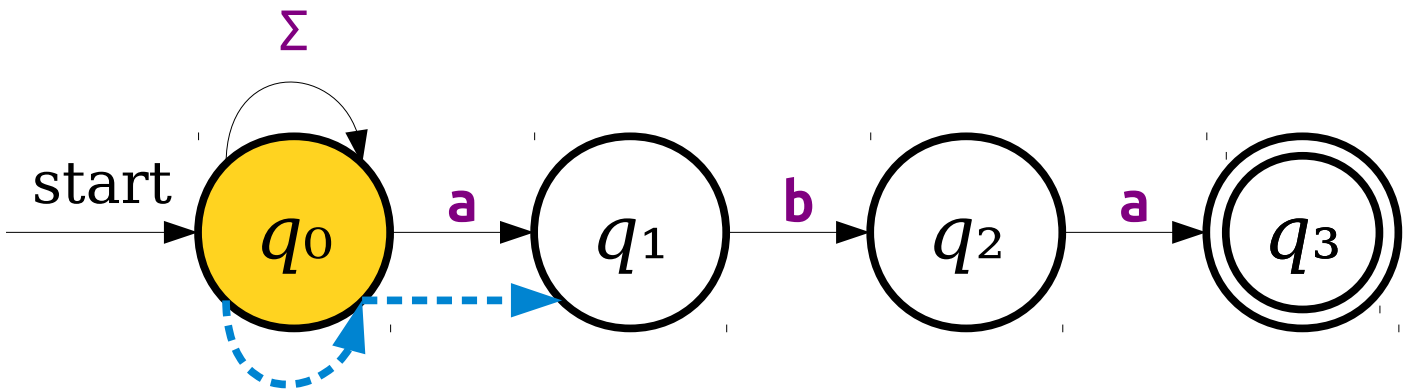


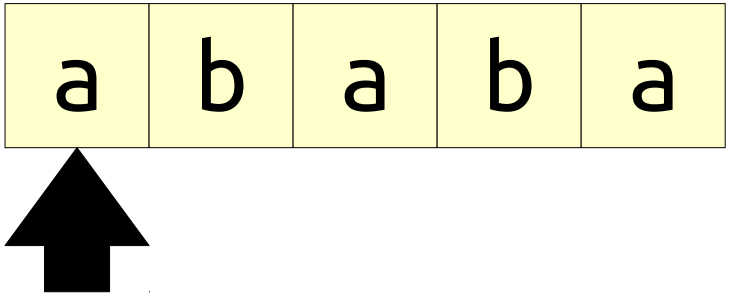

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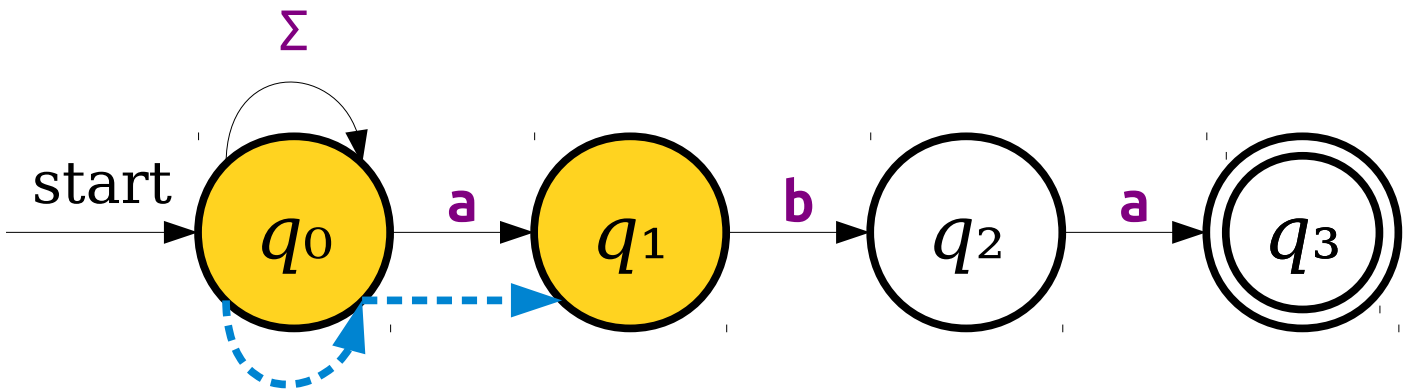


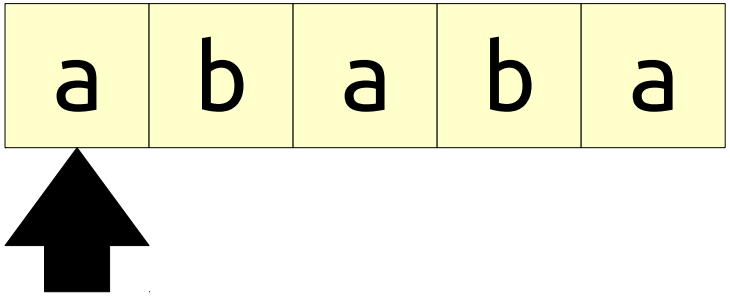

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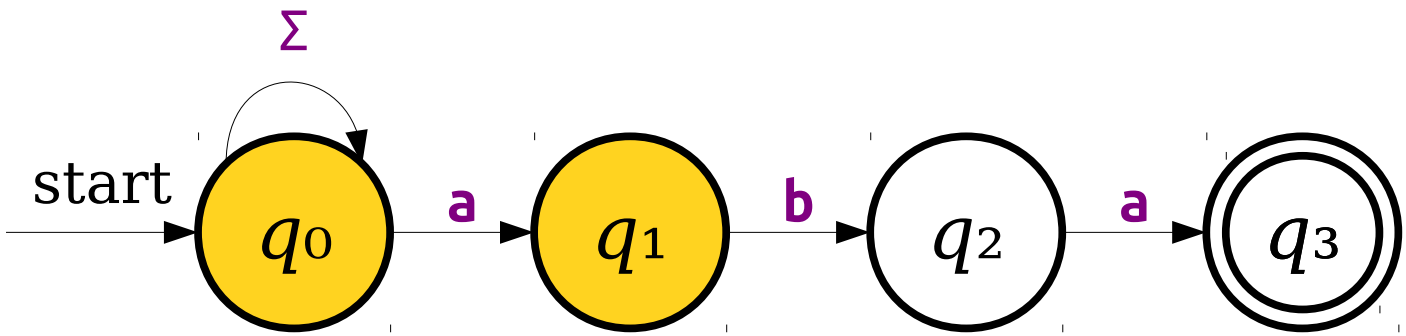


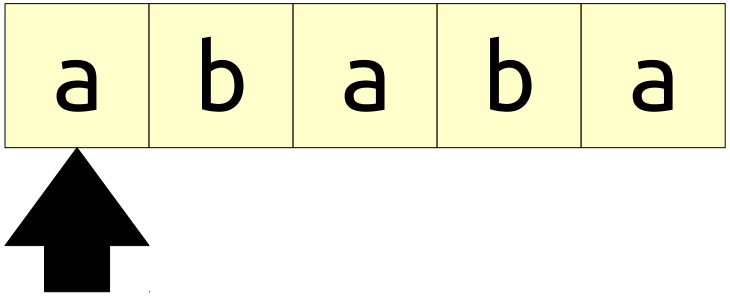

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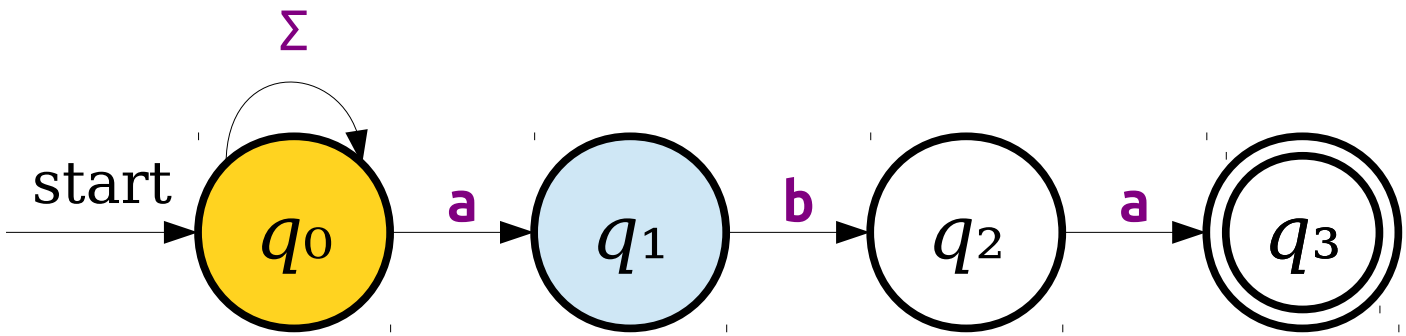


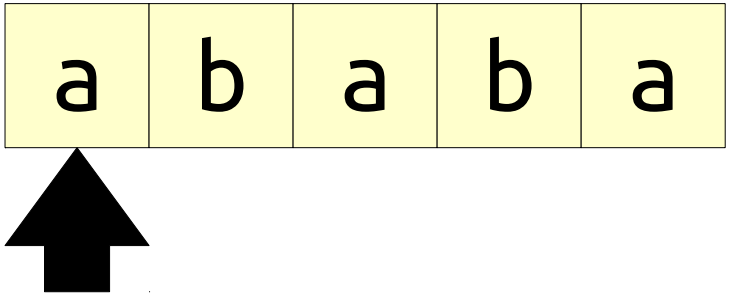

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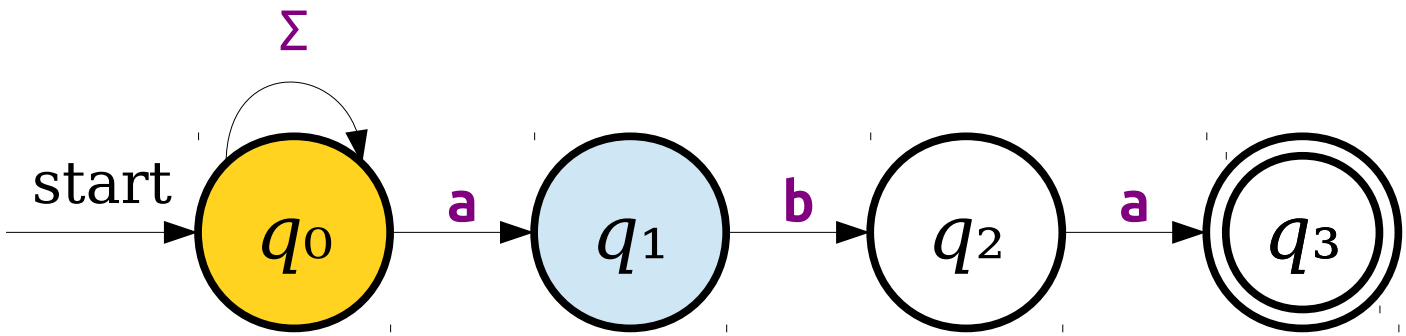


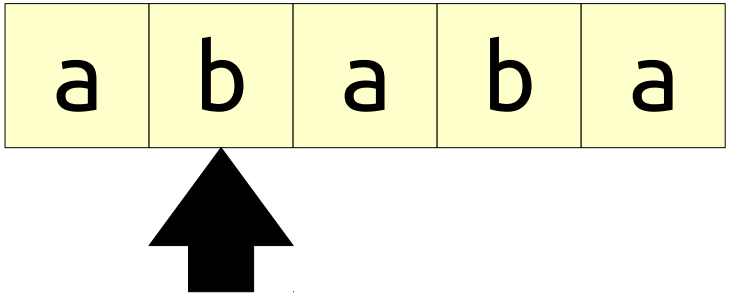

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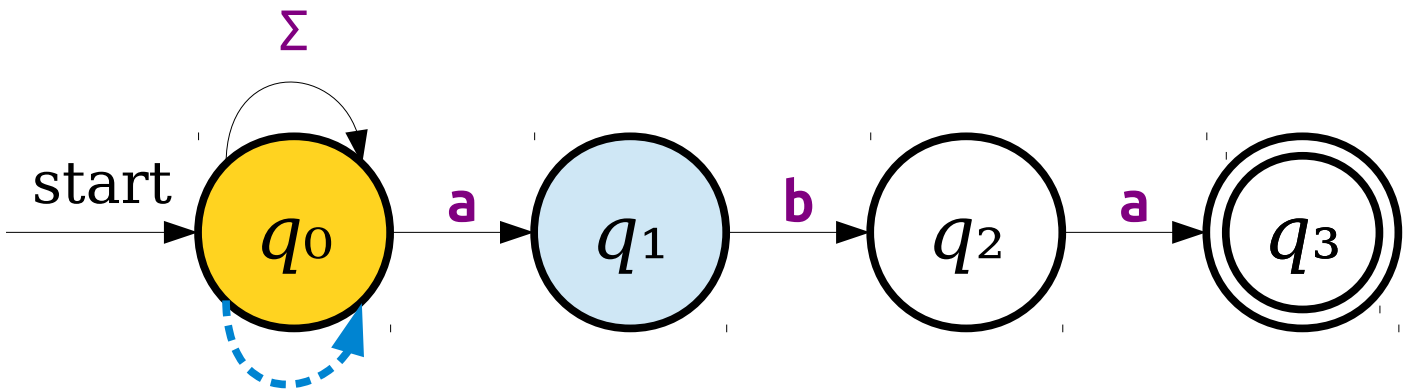


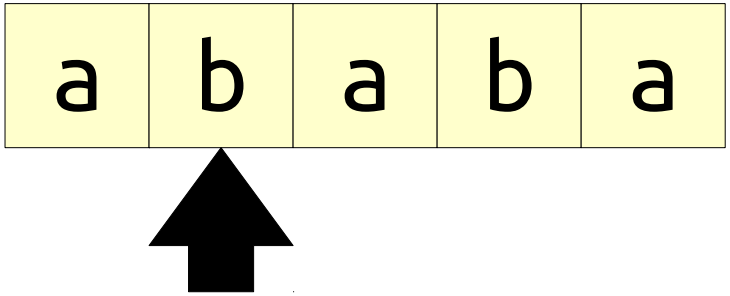

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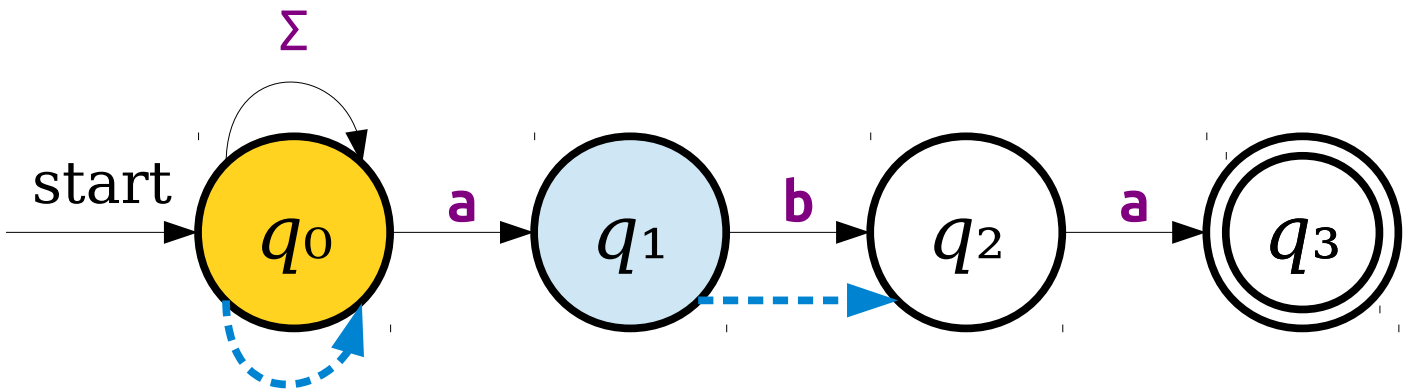


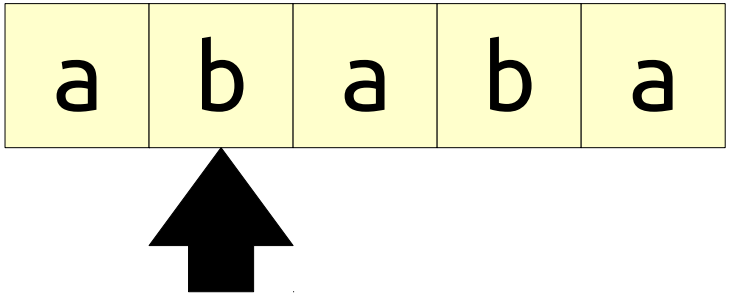

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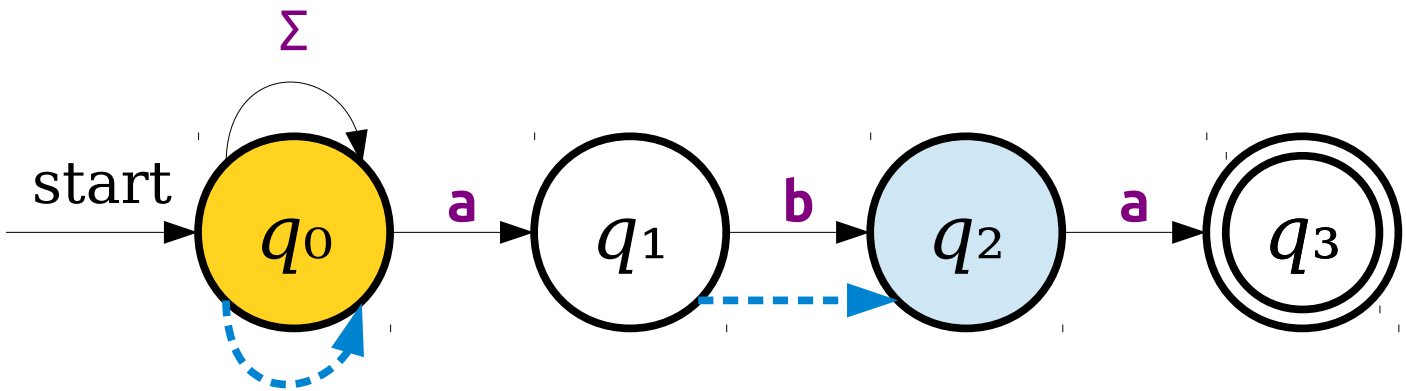


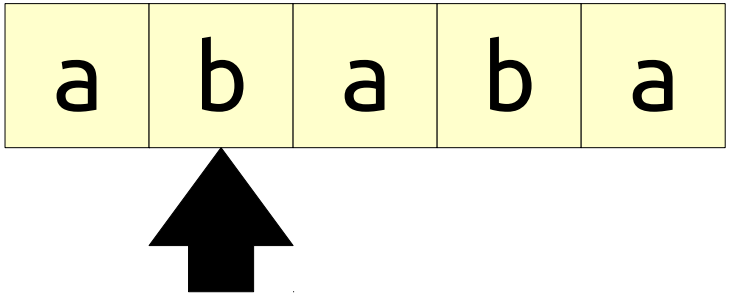

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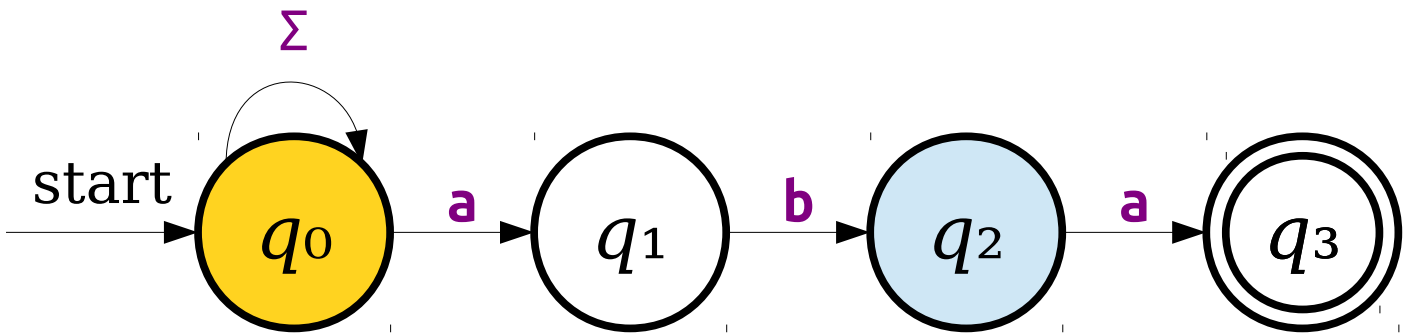


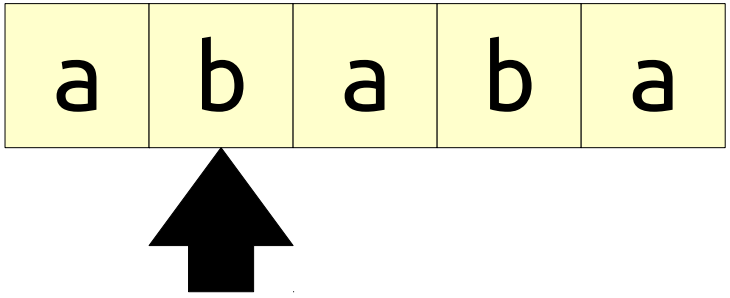

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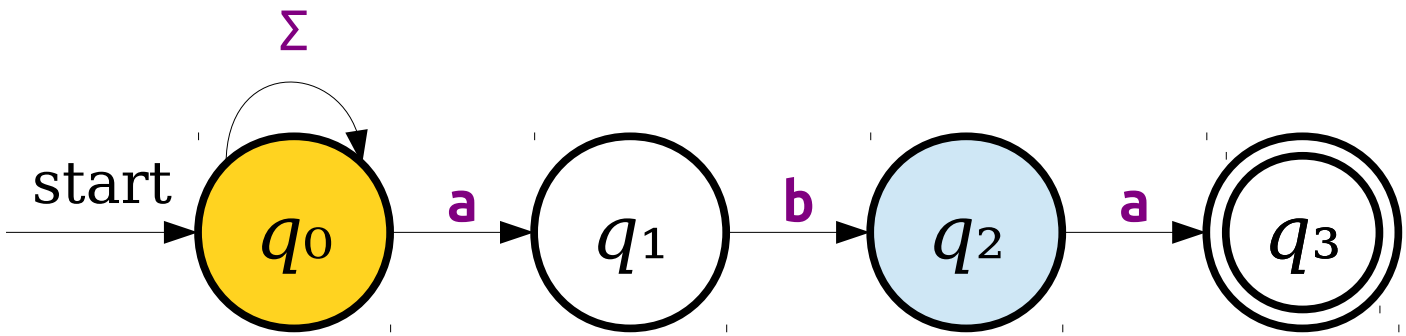


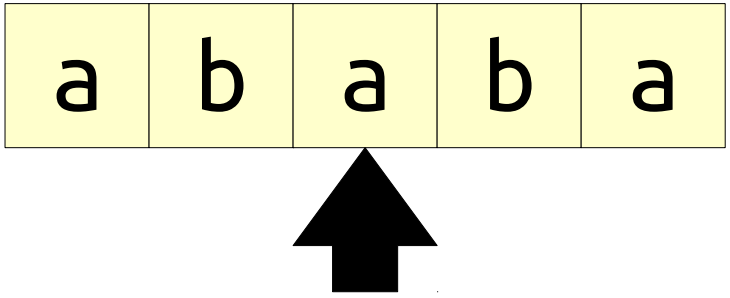

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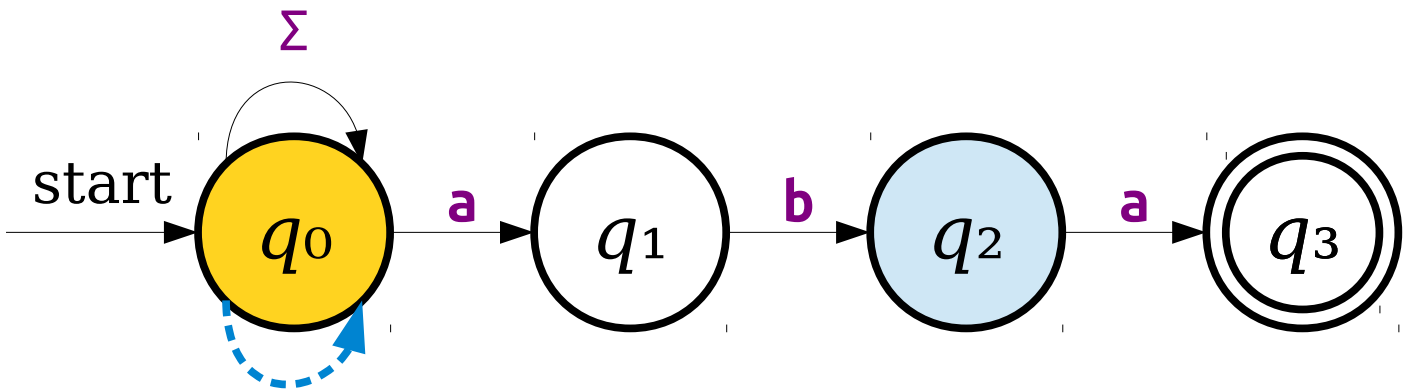


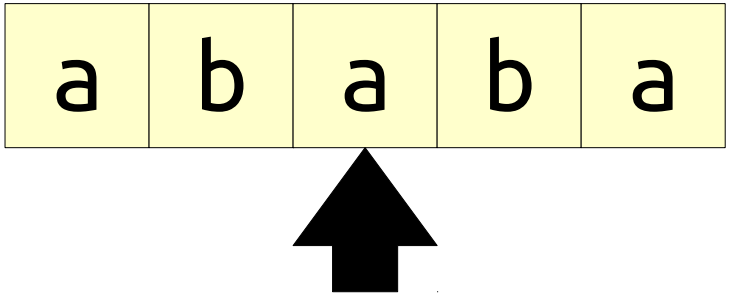

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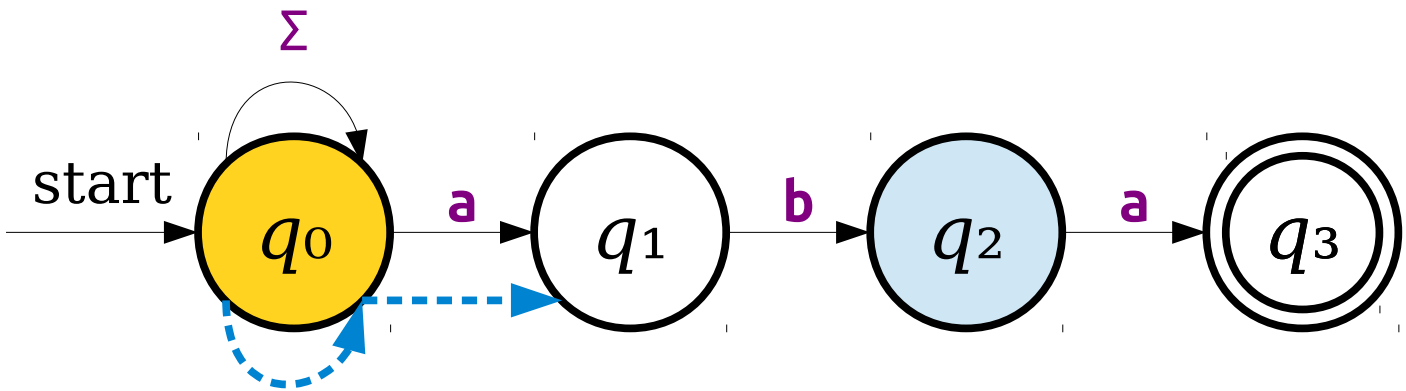


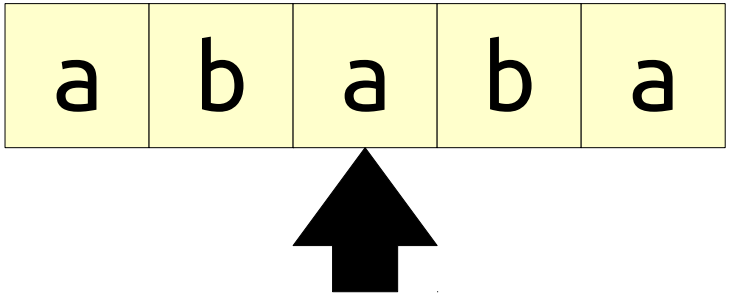

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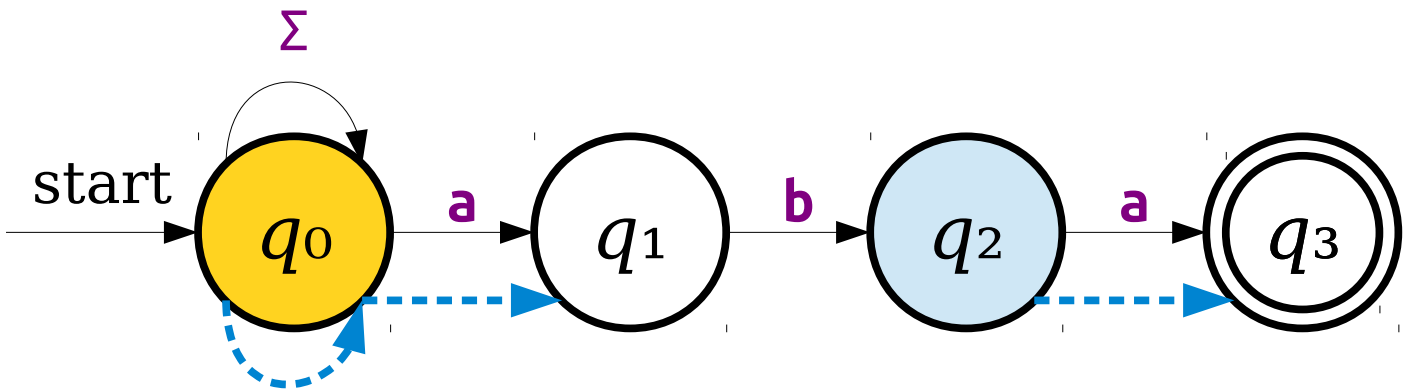


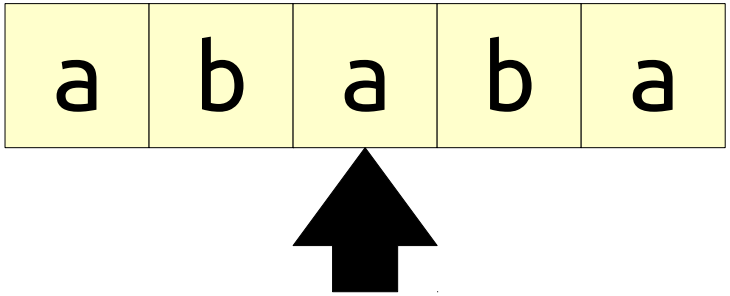

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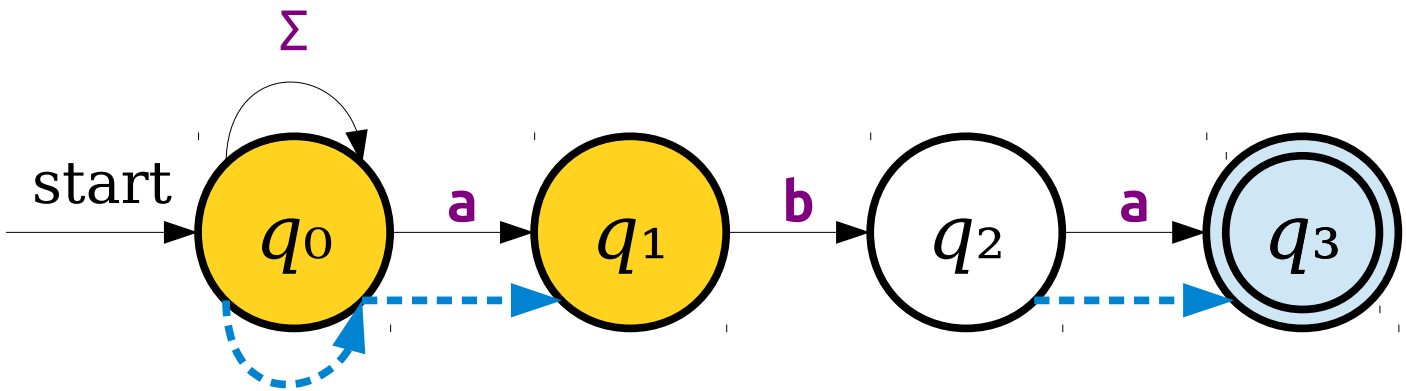


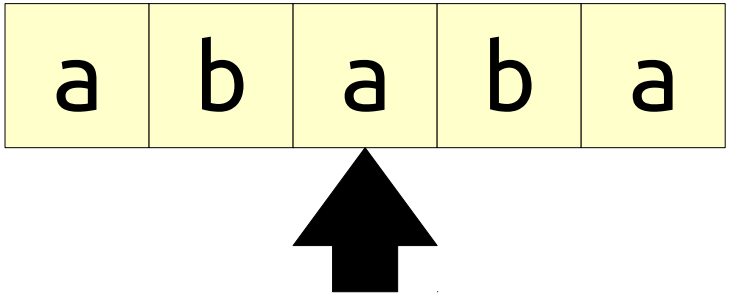

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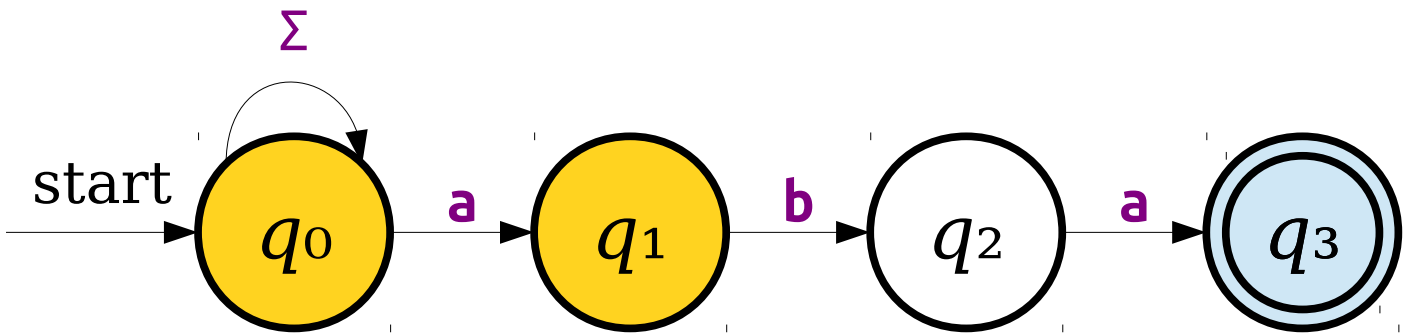


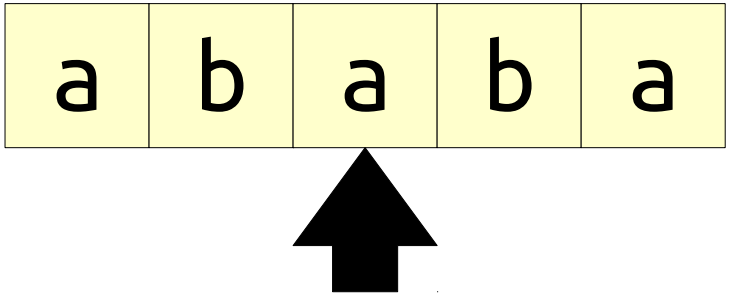

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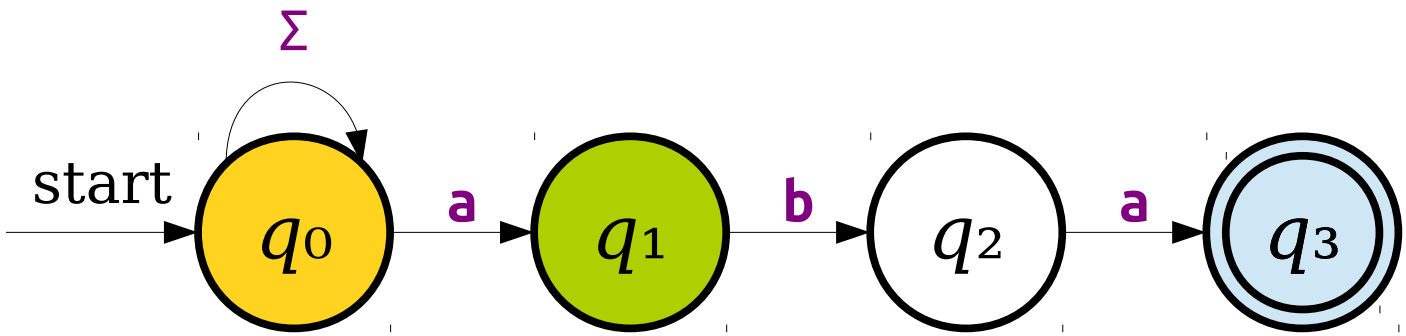


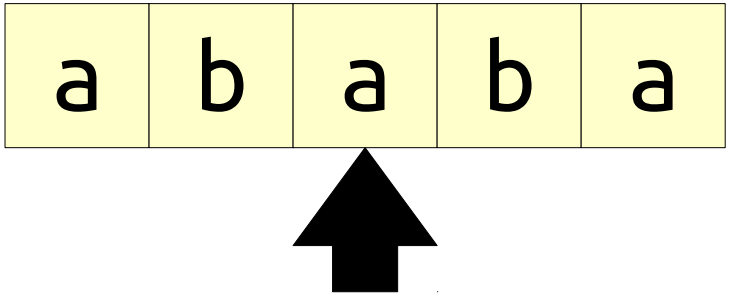

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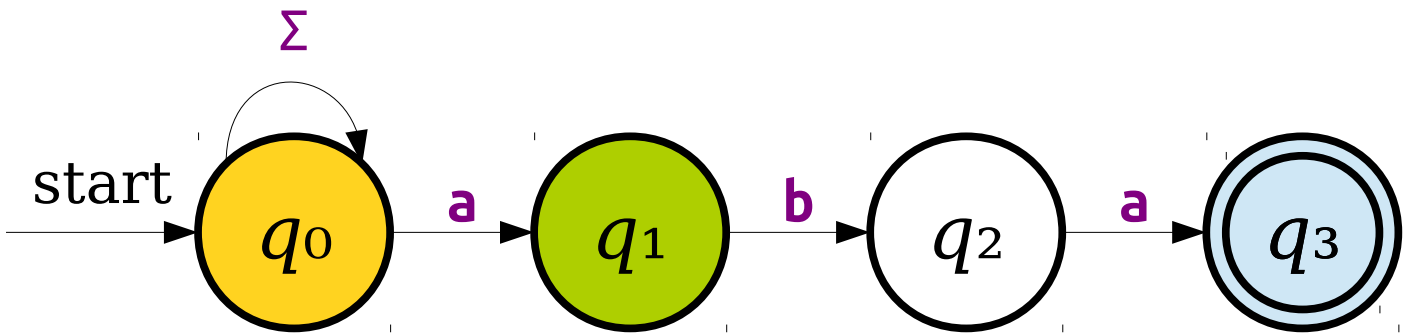


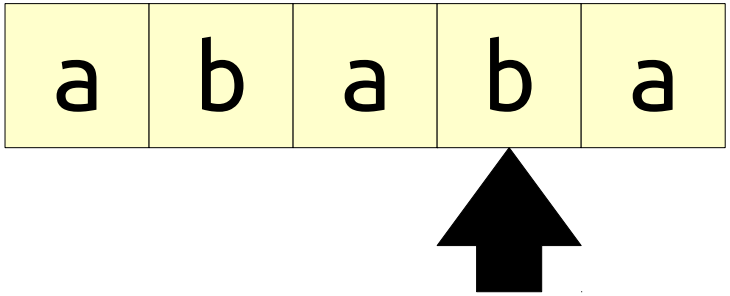

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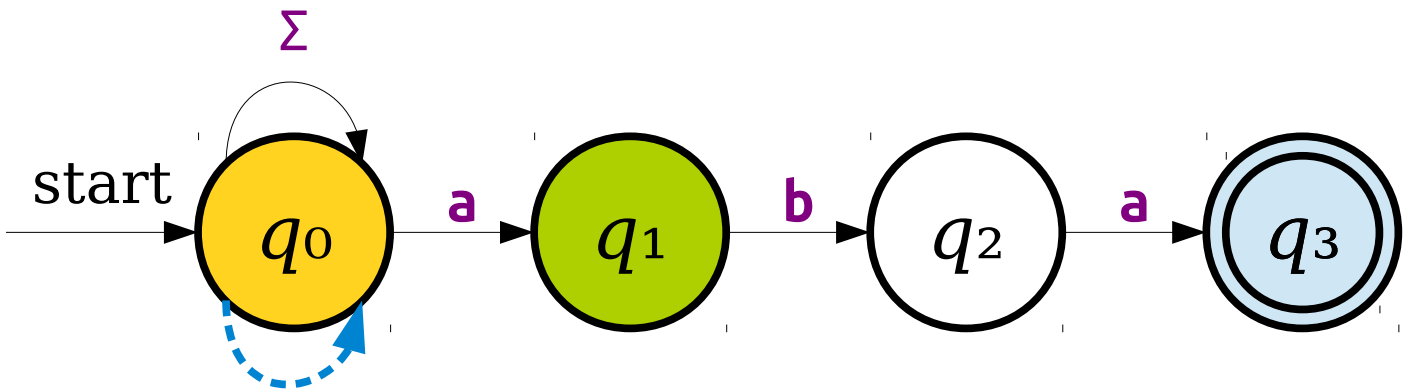


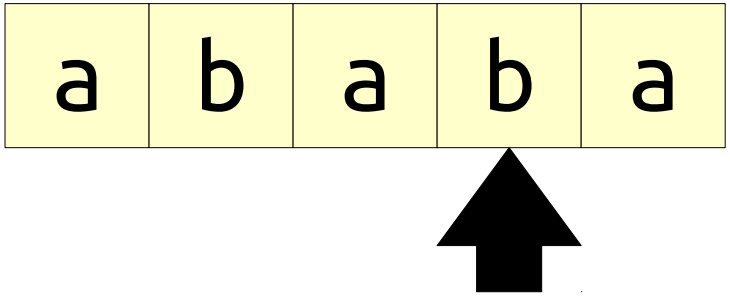

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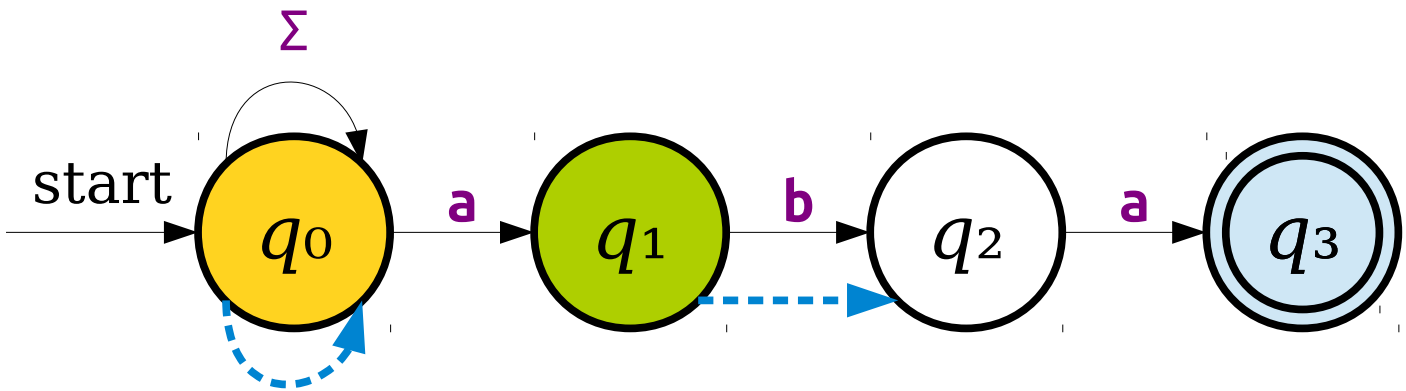


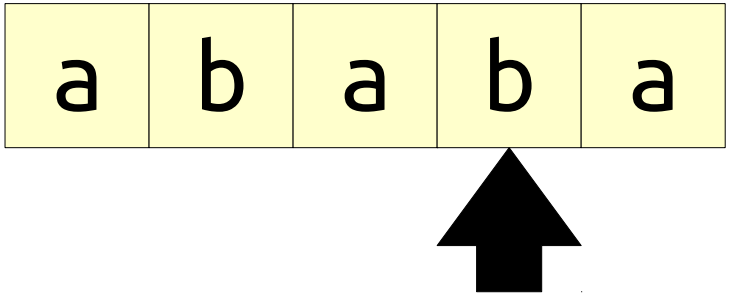

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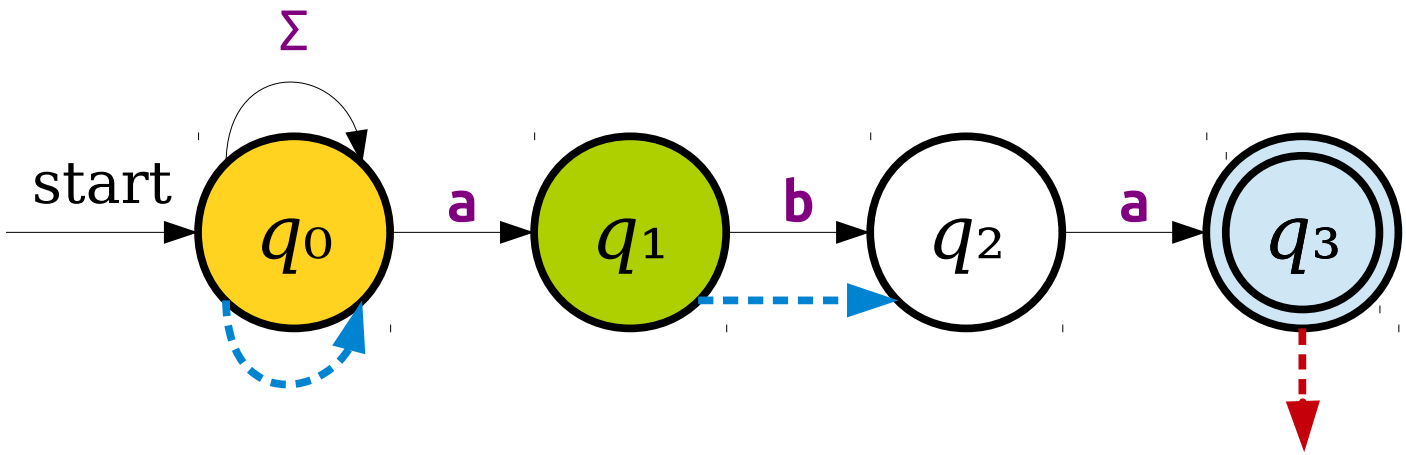


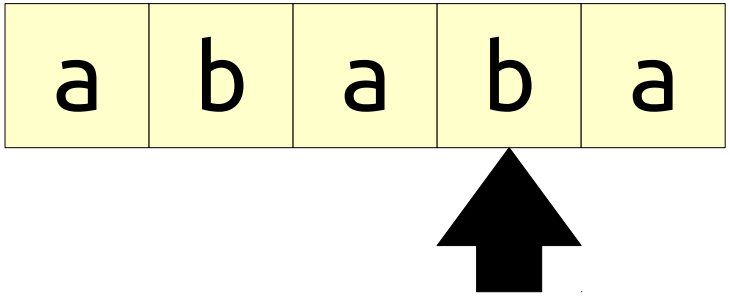

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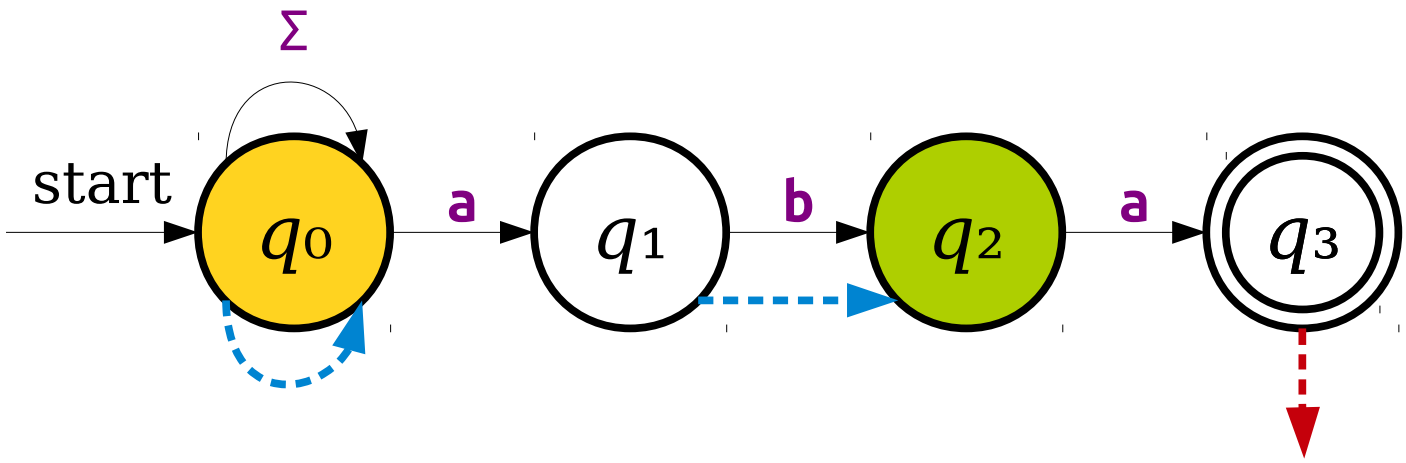


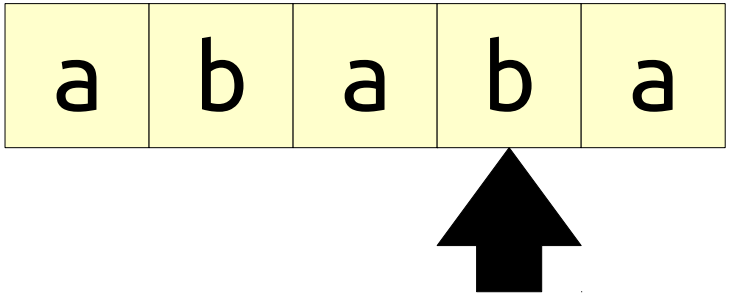

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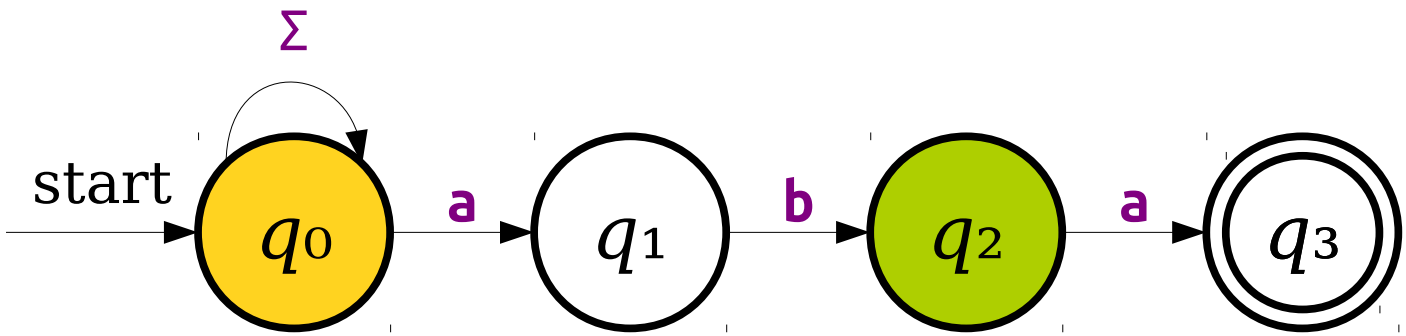
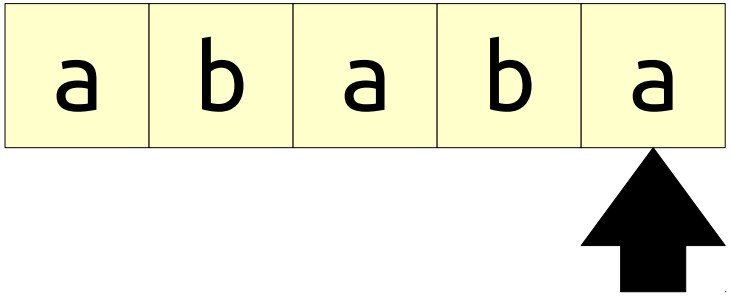
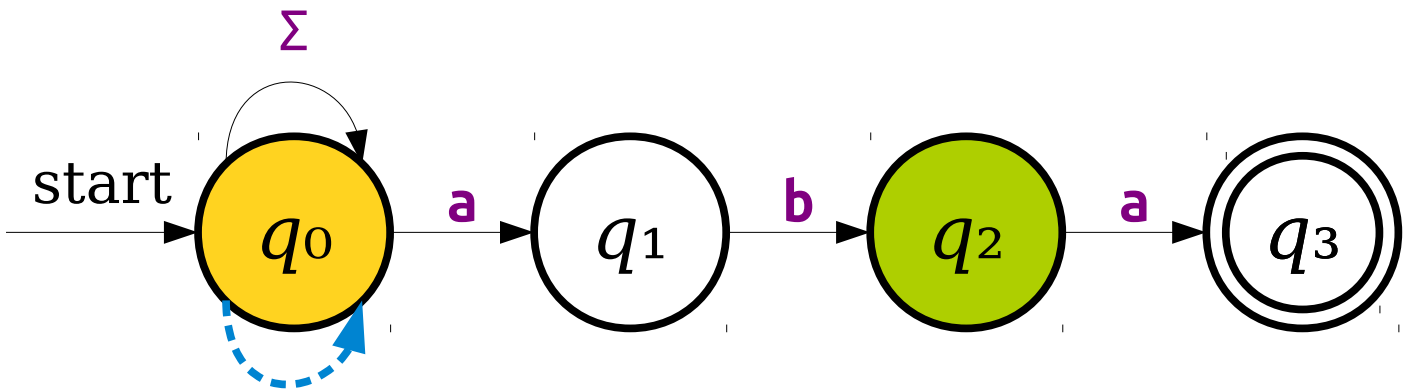


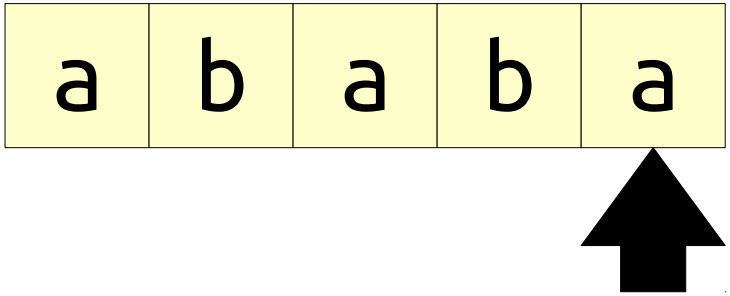




Using the massive parallelism intuition, if we are in the states $q_0$ and $q_2$, what set of states will we be in after reading the character **a**?

***Respond at pollev.com/cs103***

# Massive Parallelism



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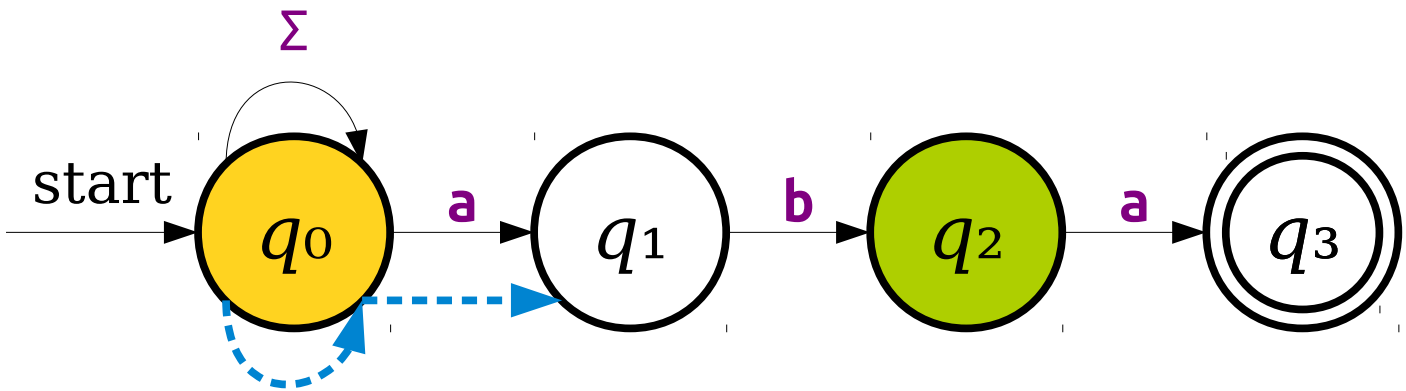


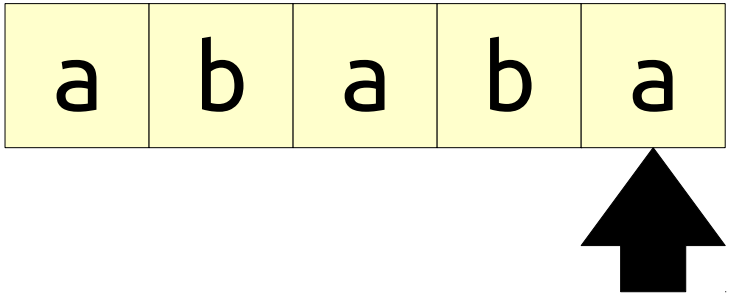

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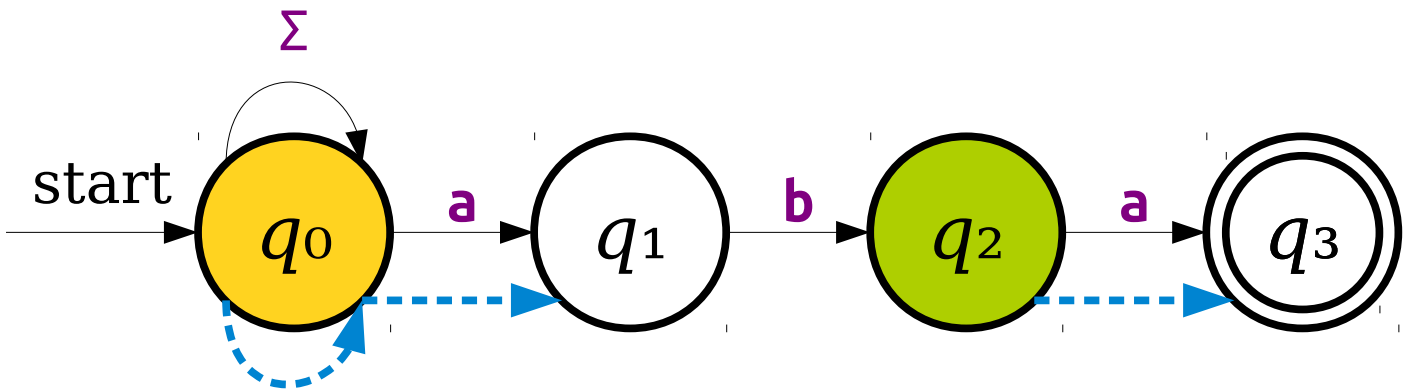


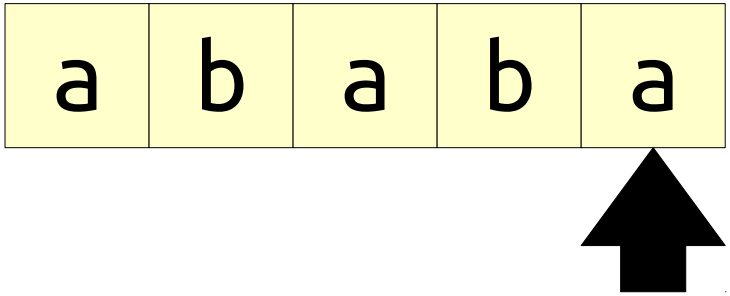

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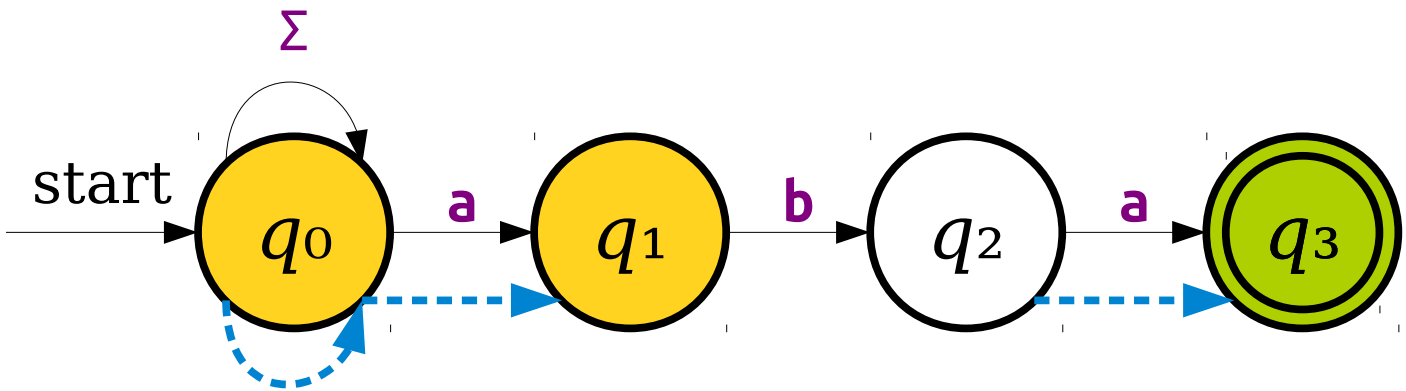


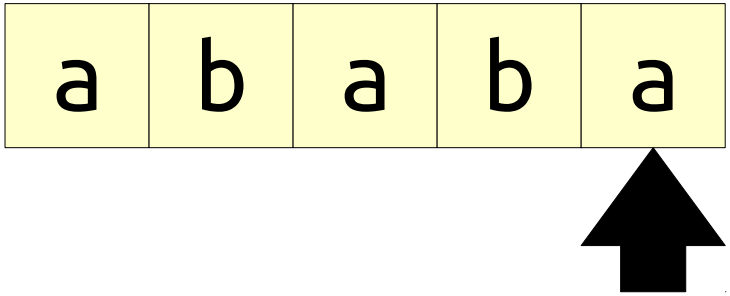

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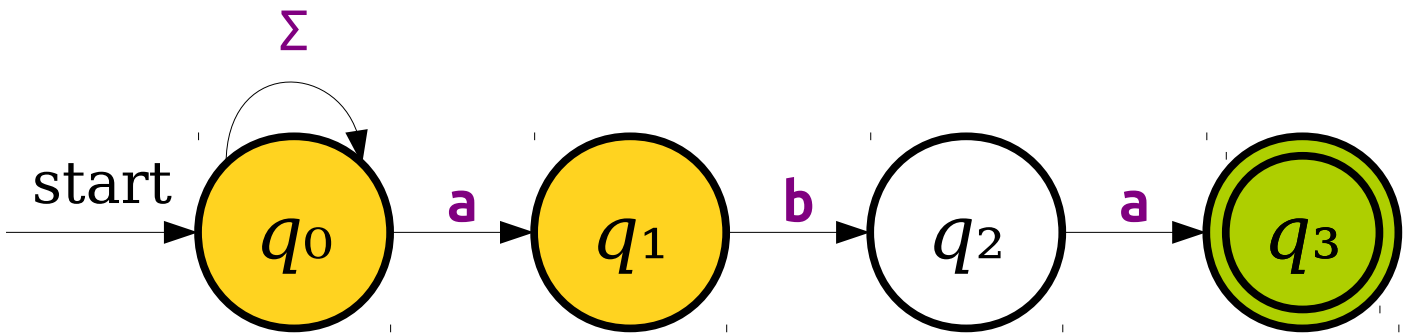
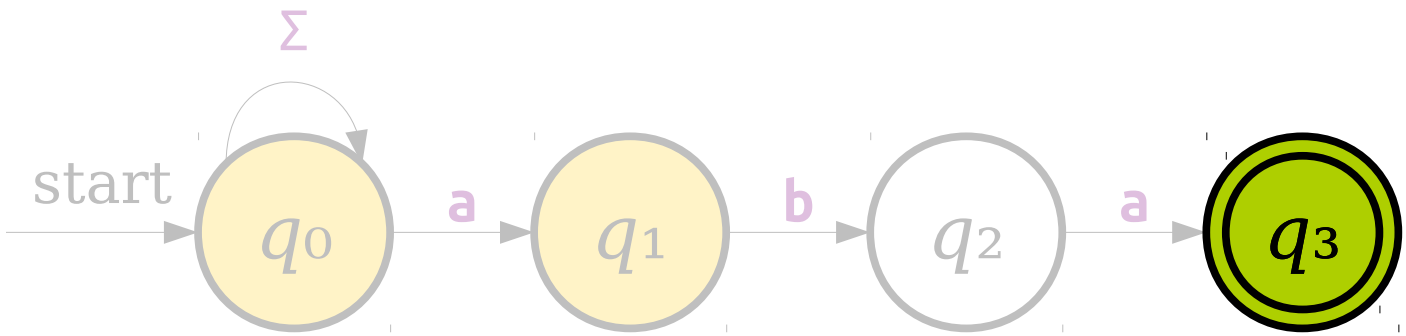



| | | | | |
|---|---|---|---|---|
| a | b | a | b | a |

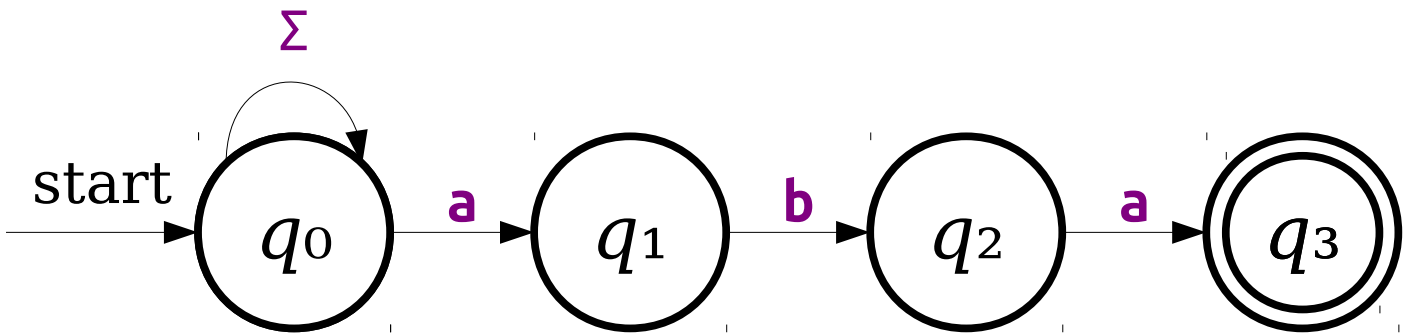
# Massive Parallelism

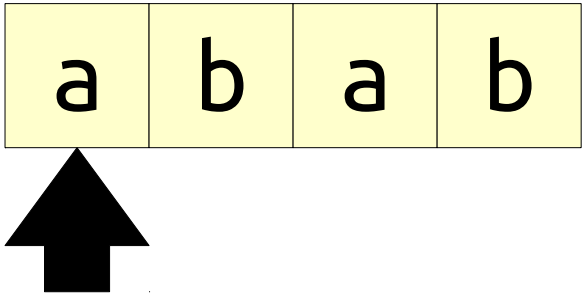



We're in at least one accepting state, so there's some path that gets us to an accepting state.

| | | | | |
|---|---|---|---|---|
| a | b | a | b | a |

# Massive Parallelism



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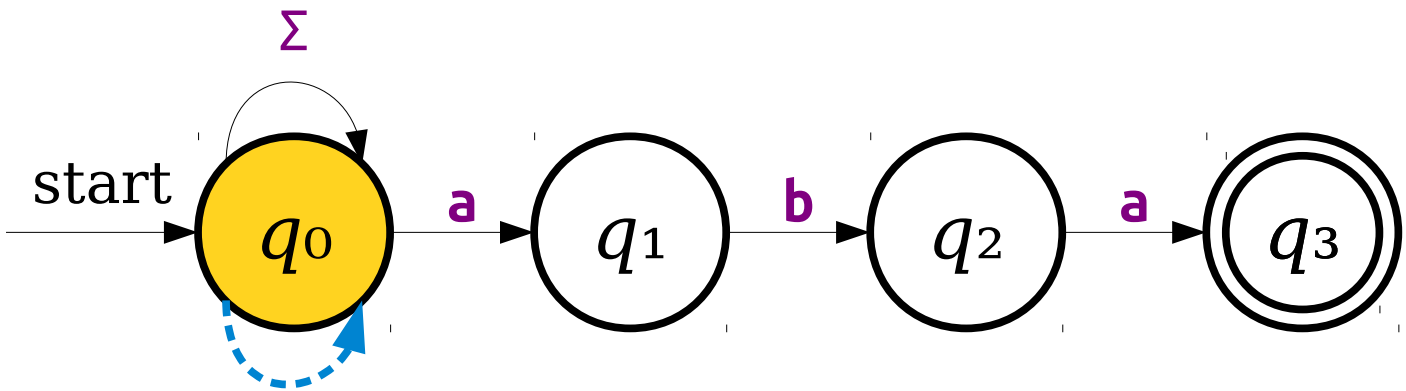


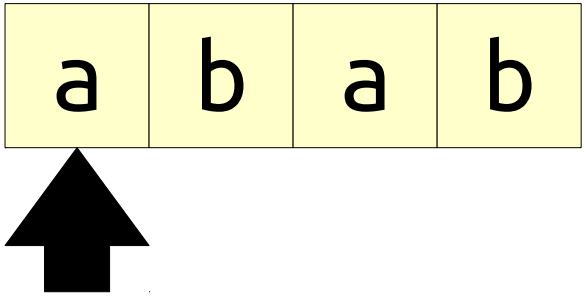

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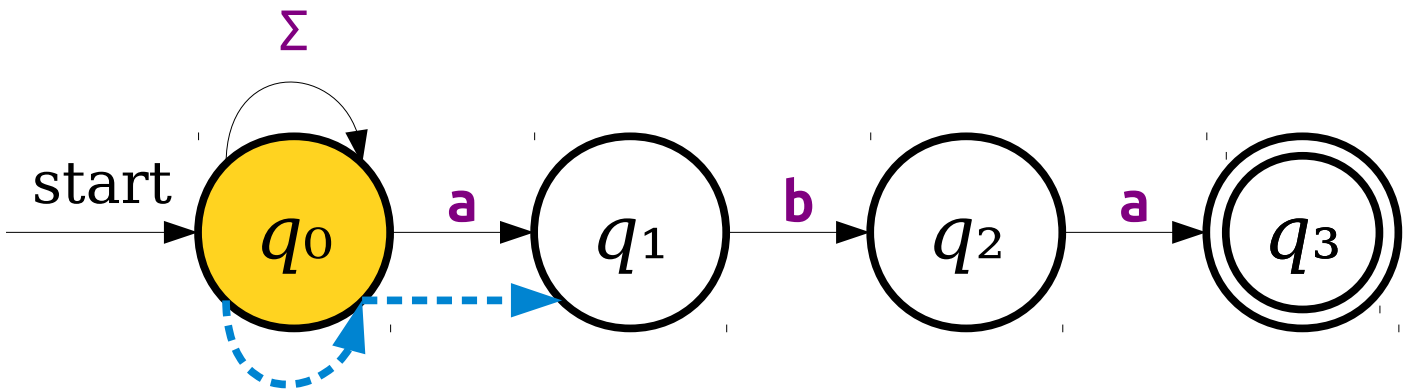


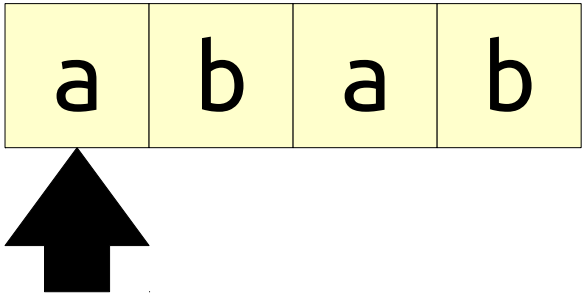

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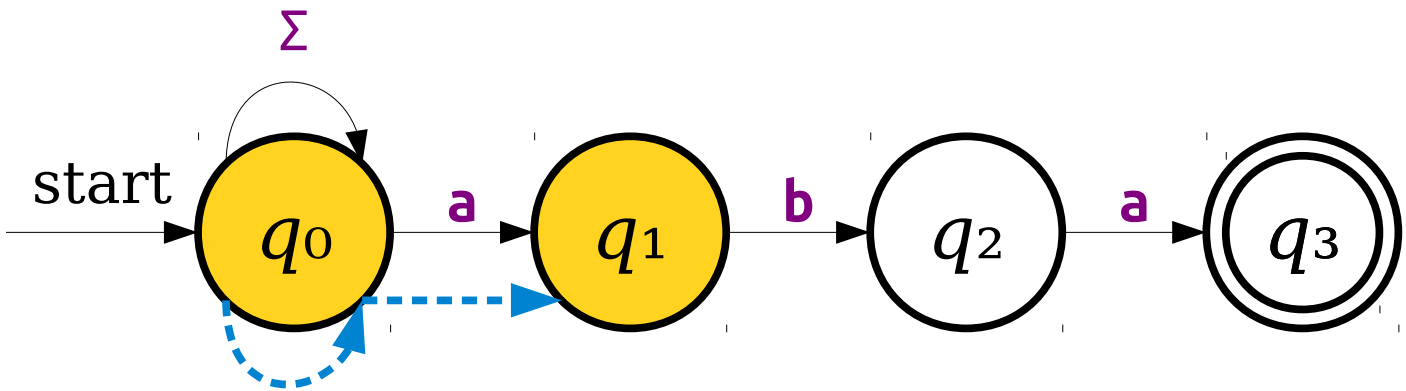


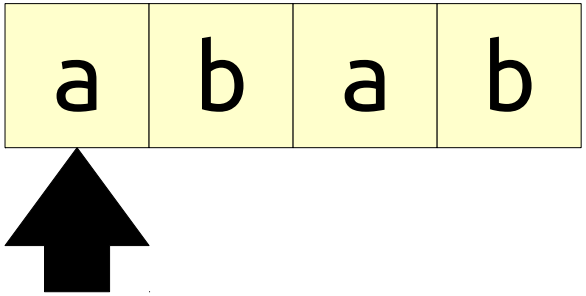

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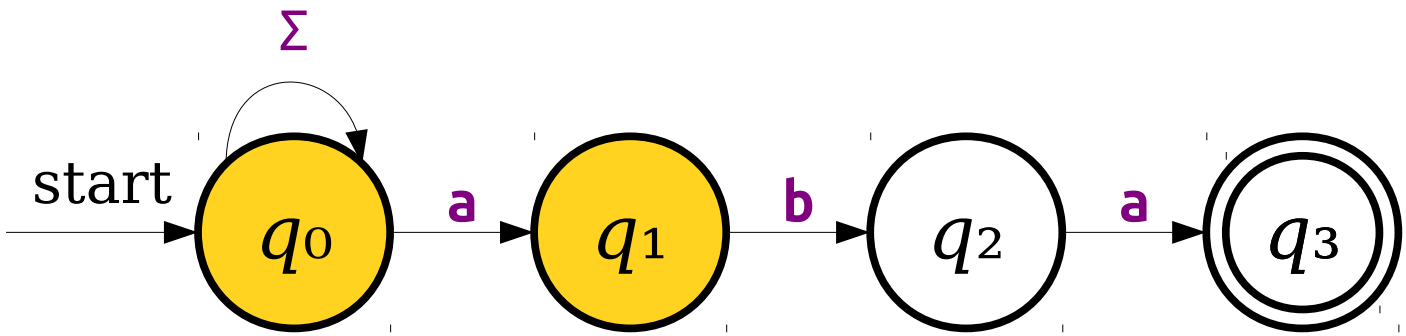


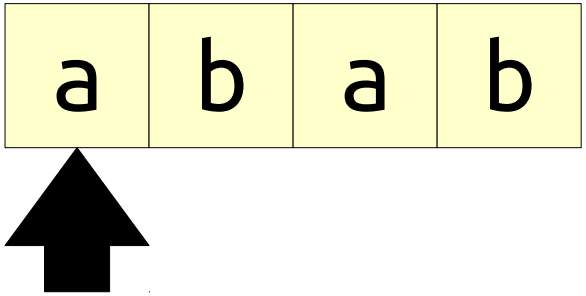

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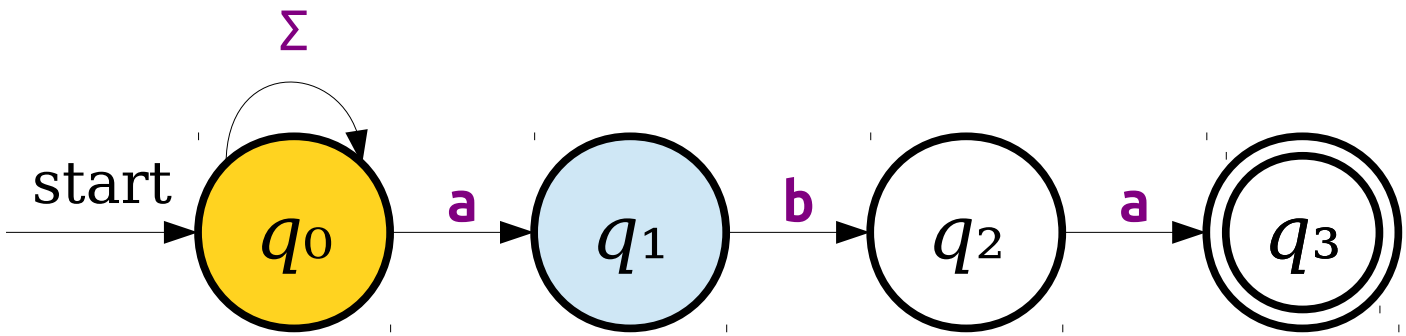


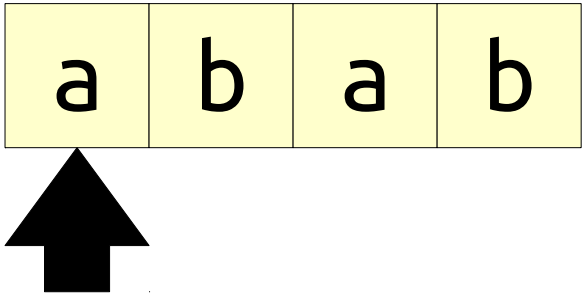

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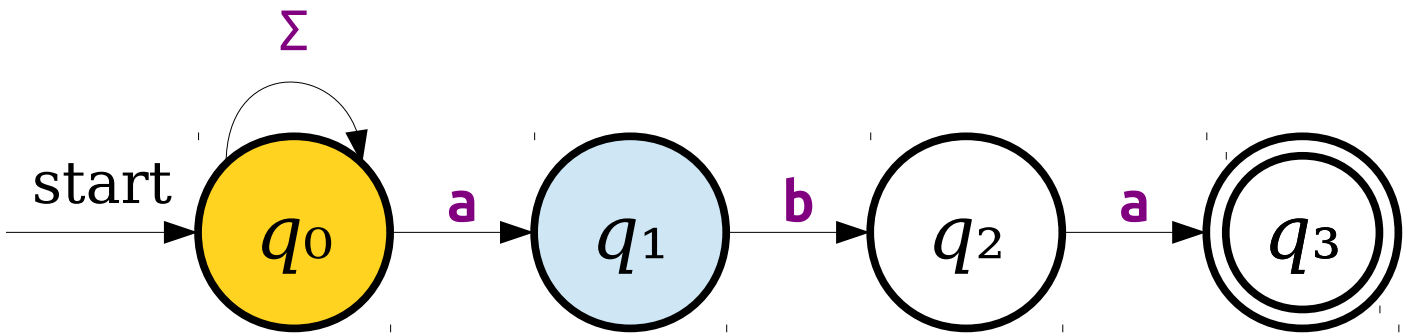


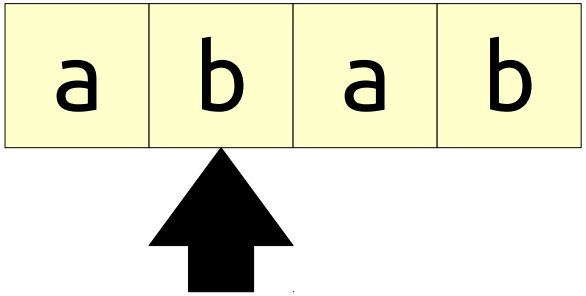

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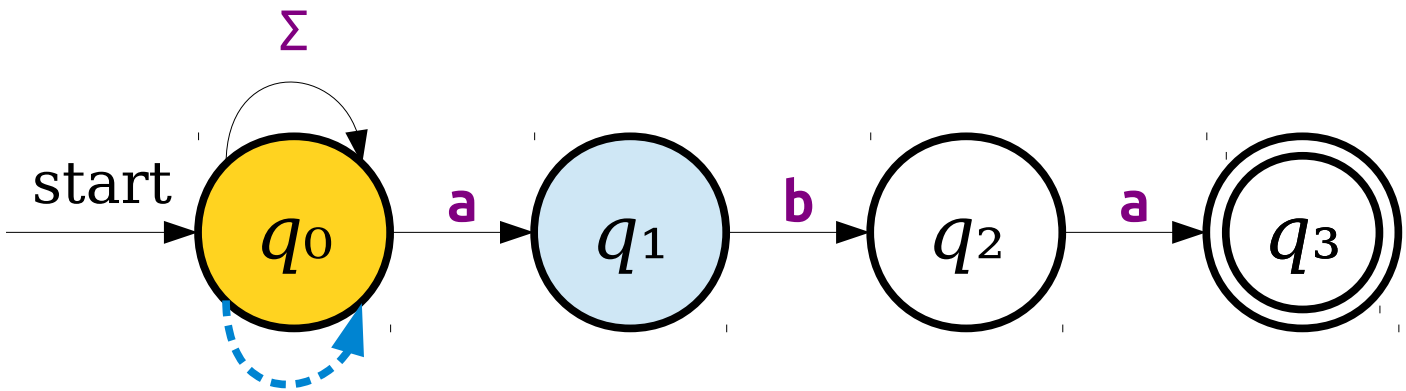


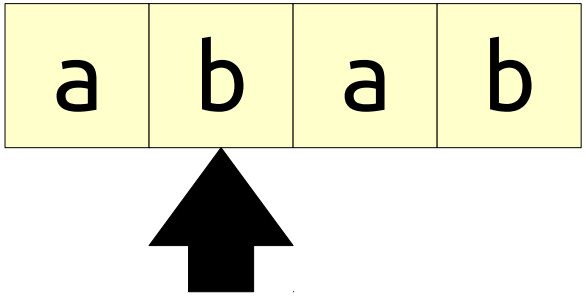

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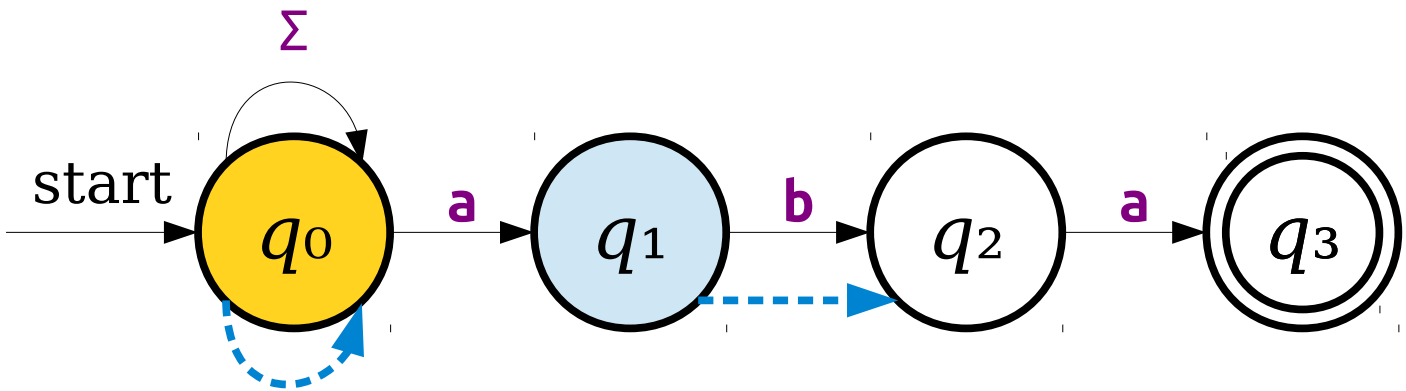


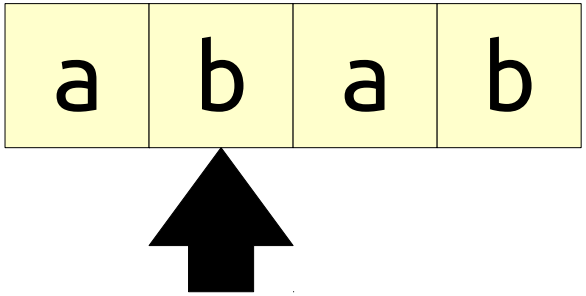

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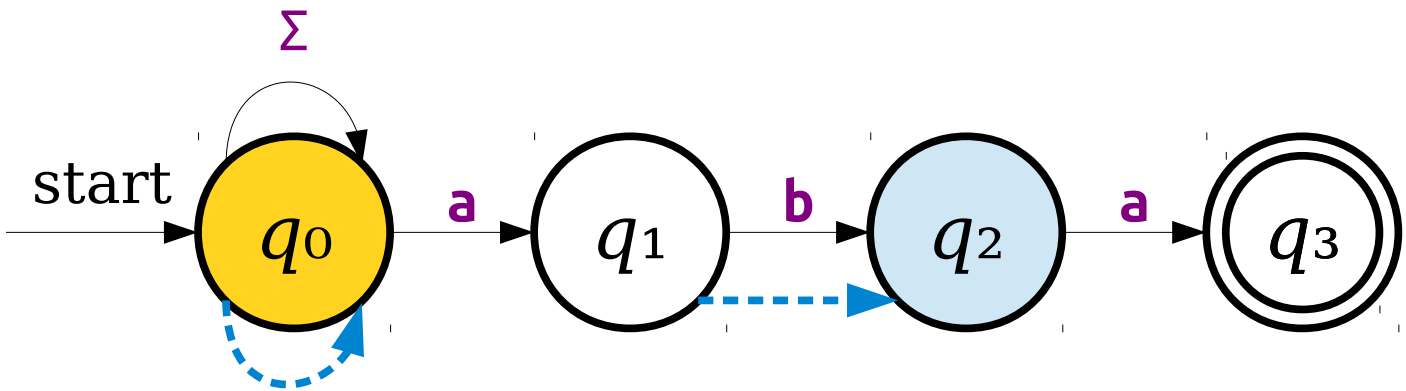


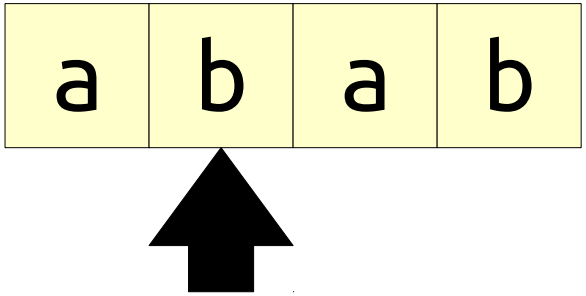

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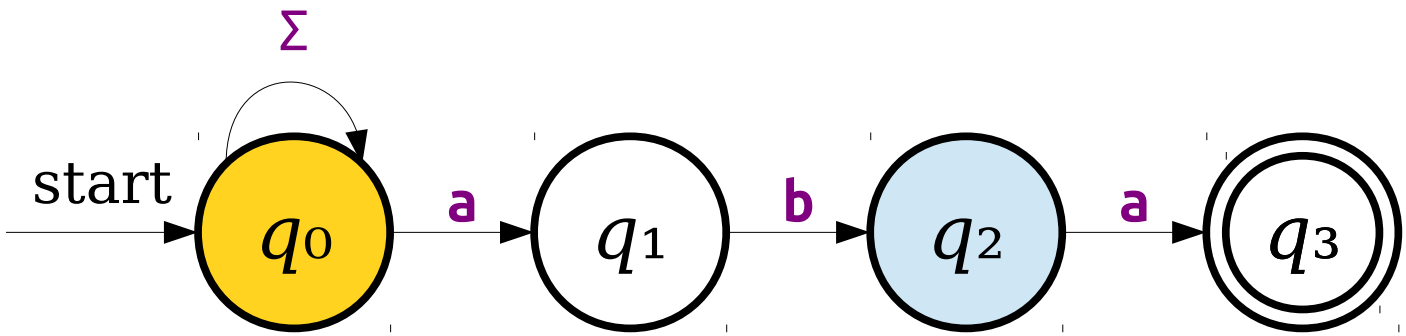


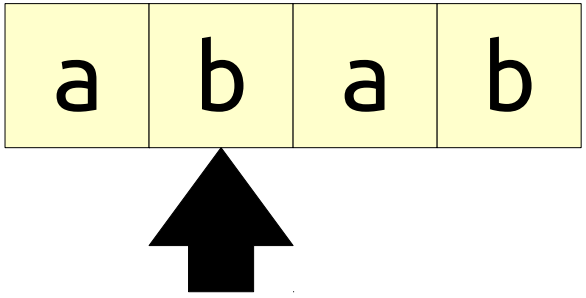

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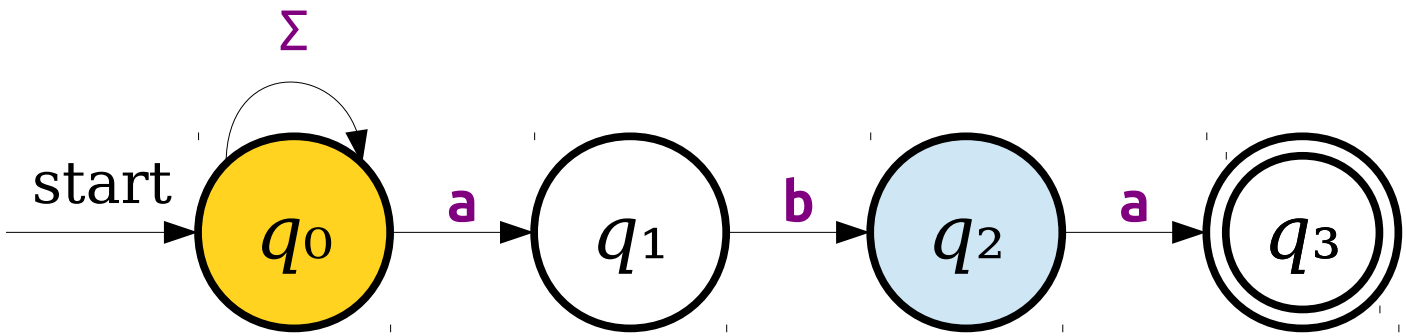


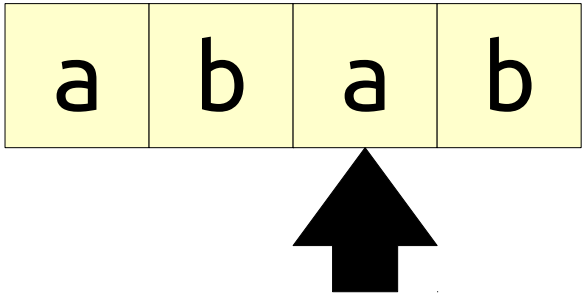

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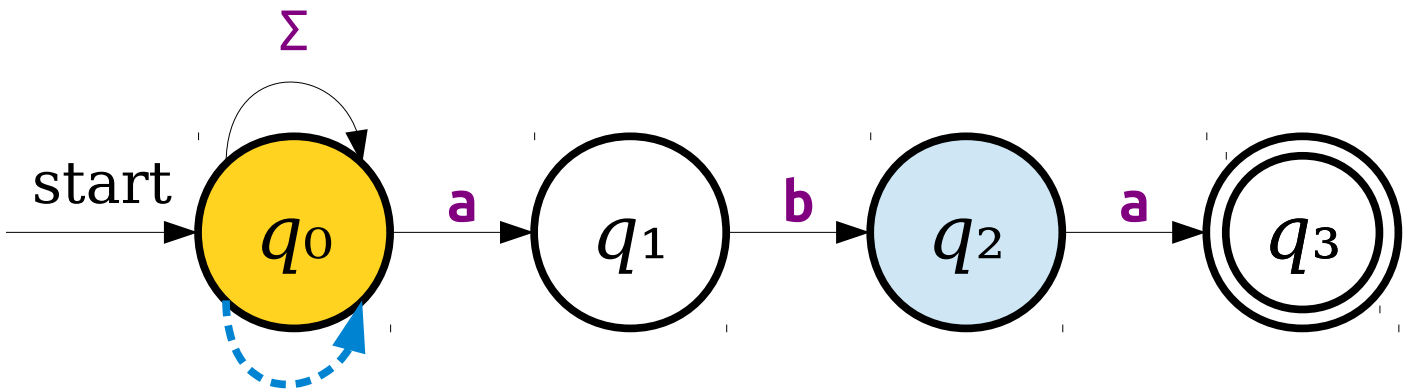
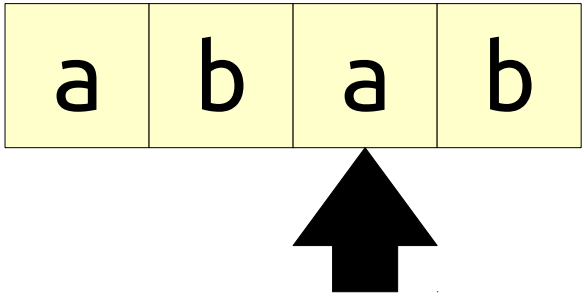
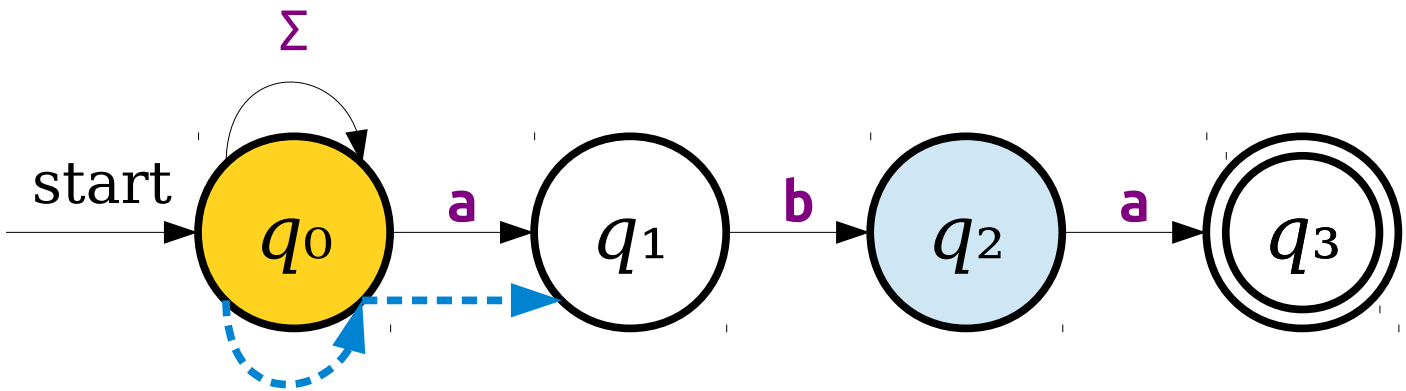
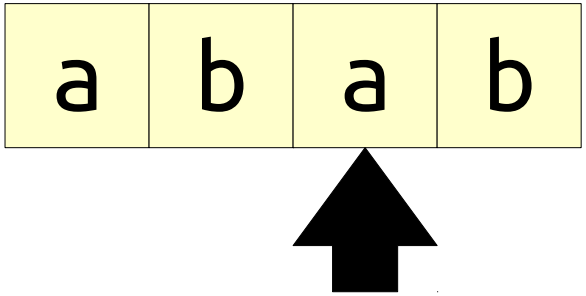


# Massive Parallelism



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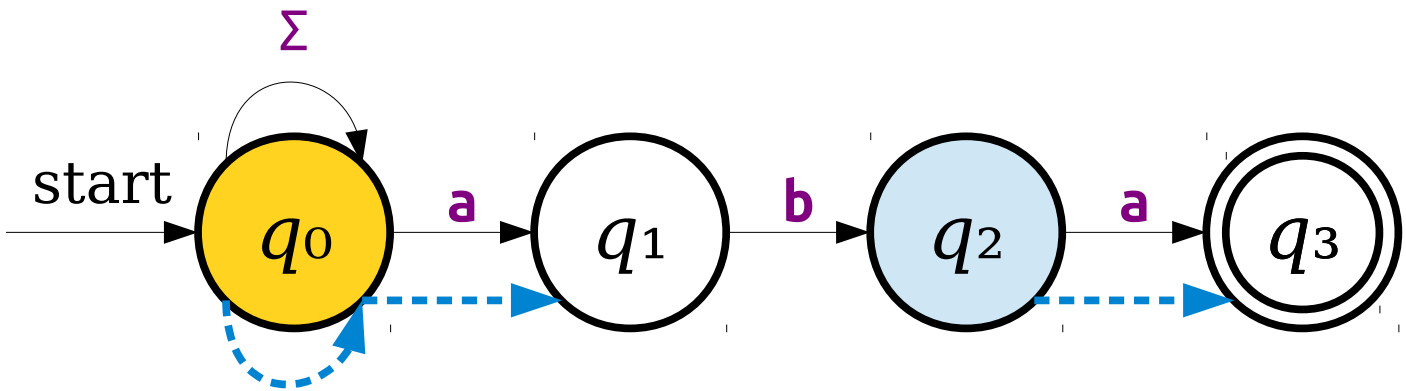


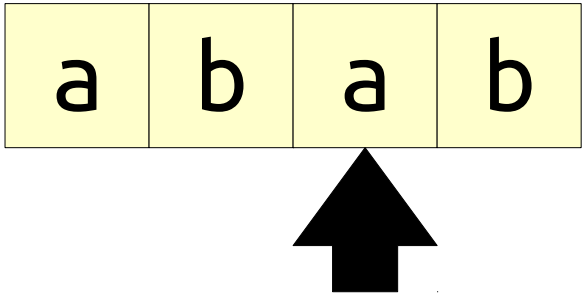

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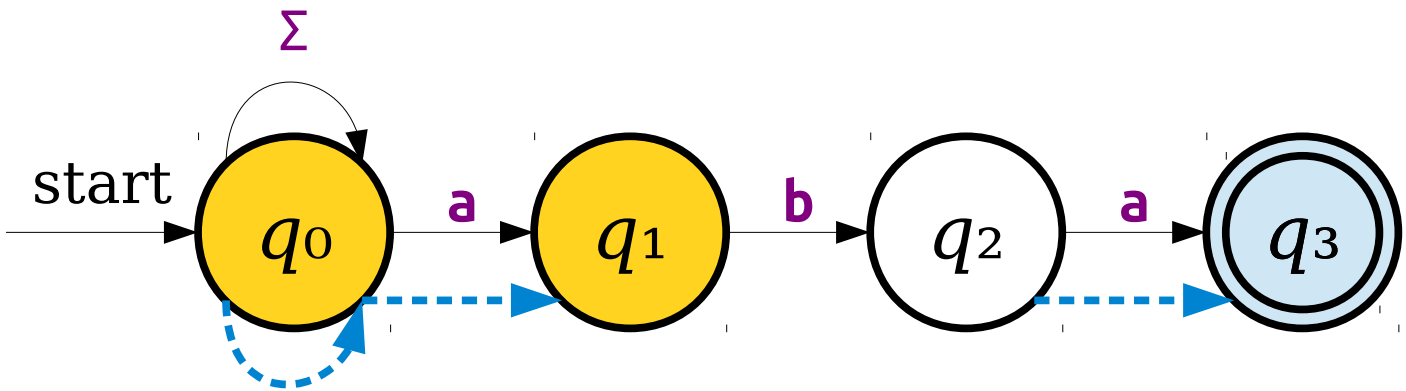


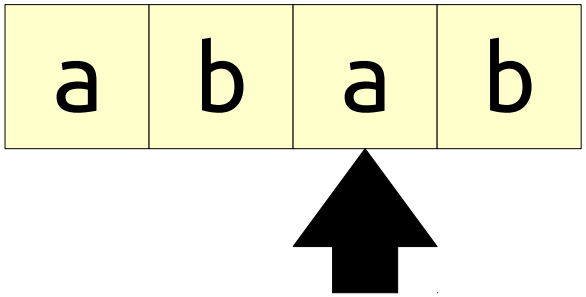

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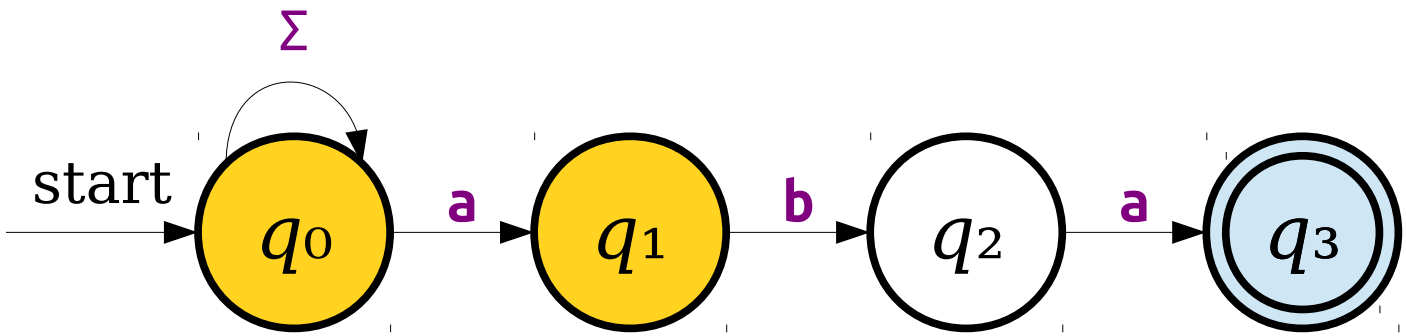


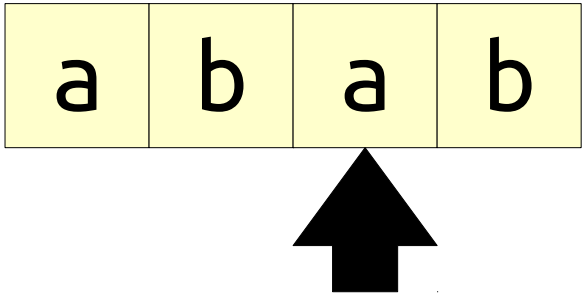

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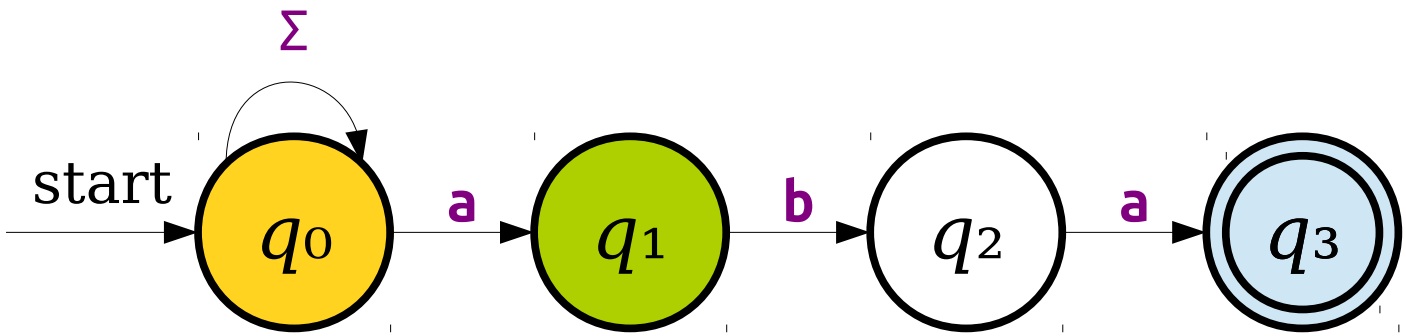


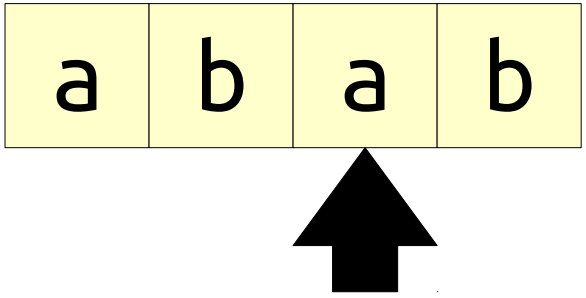

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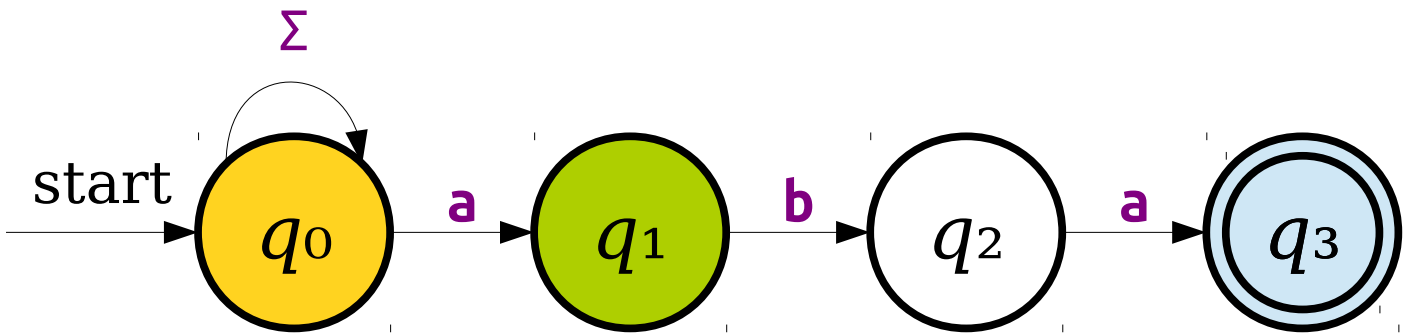


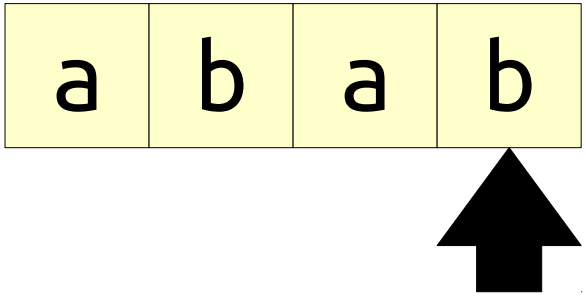

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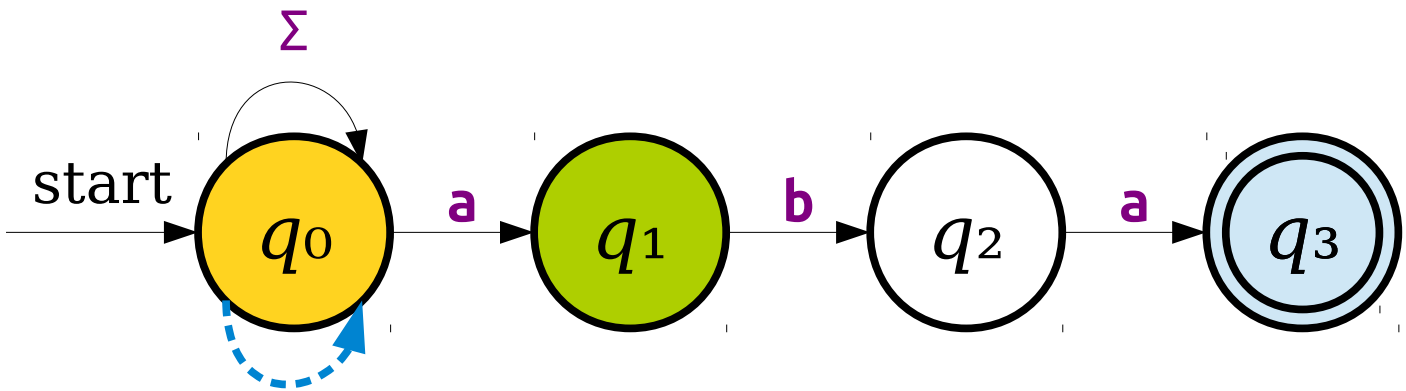


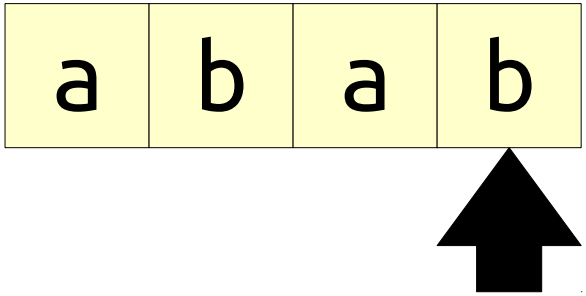

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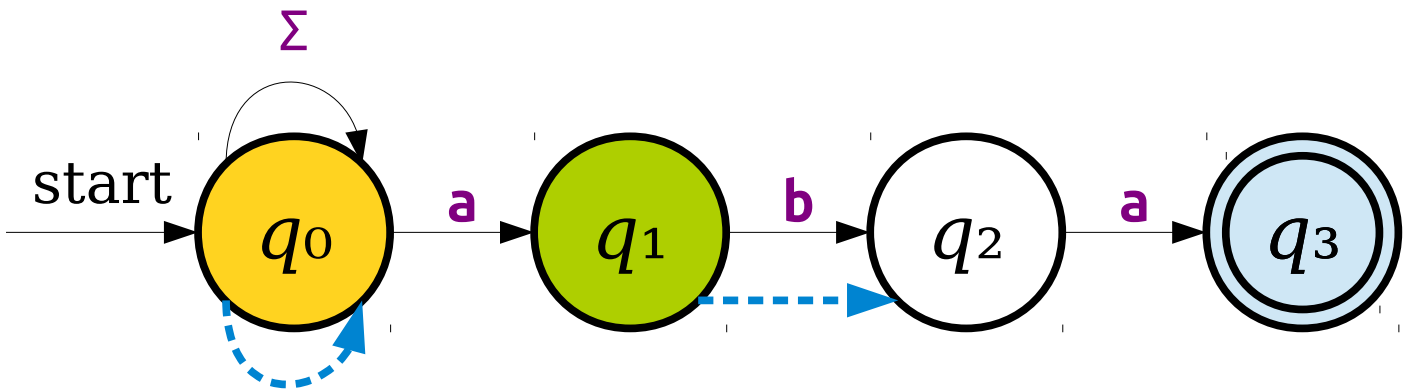


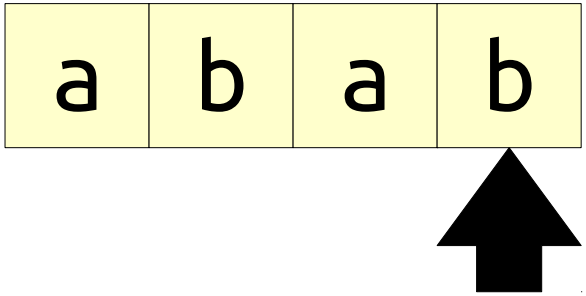

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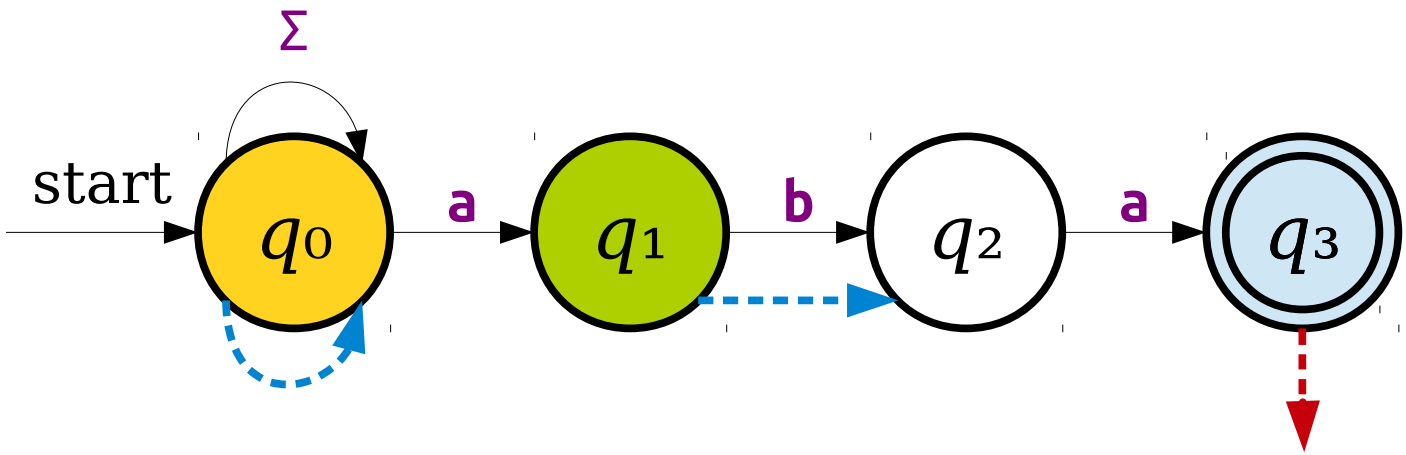


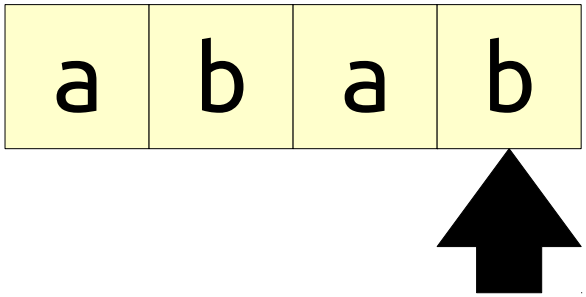

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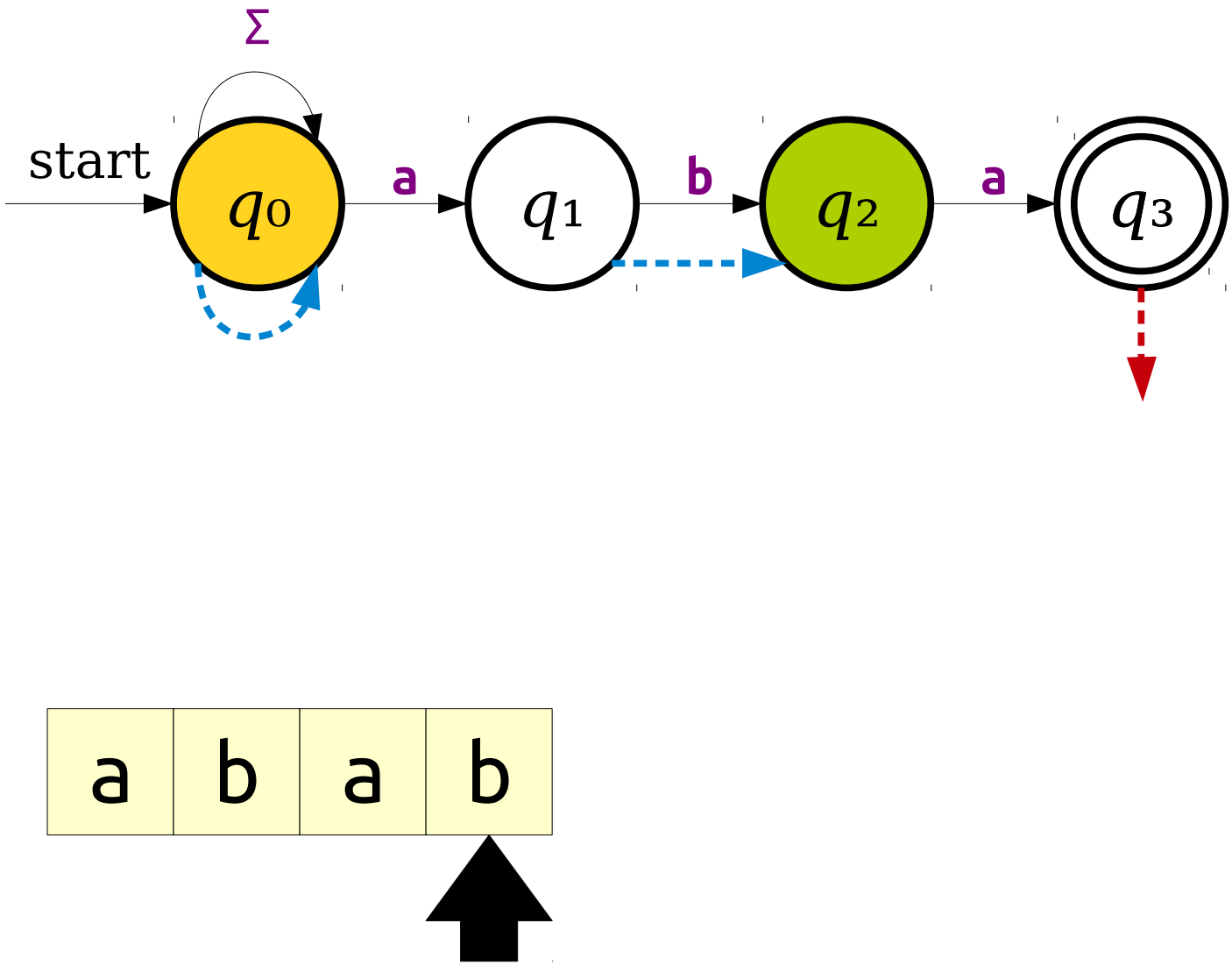

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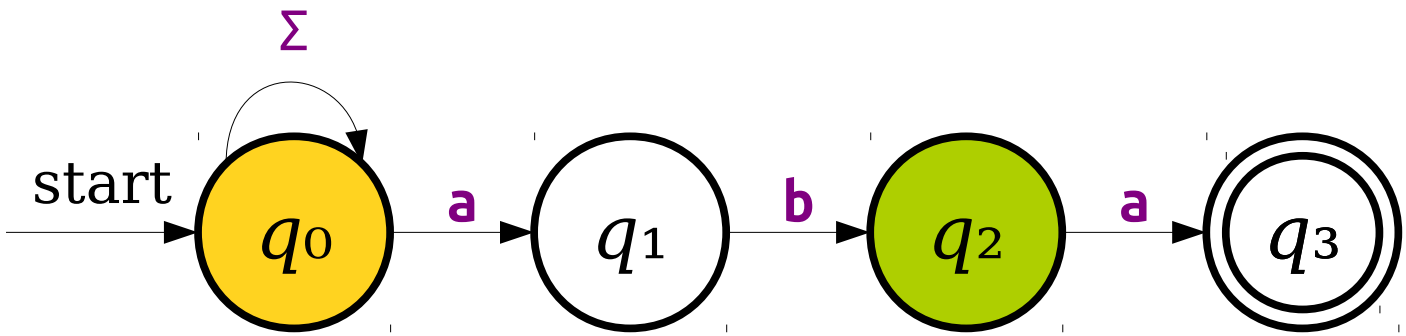
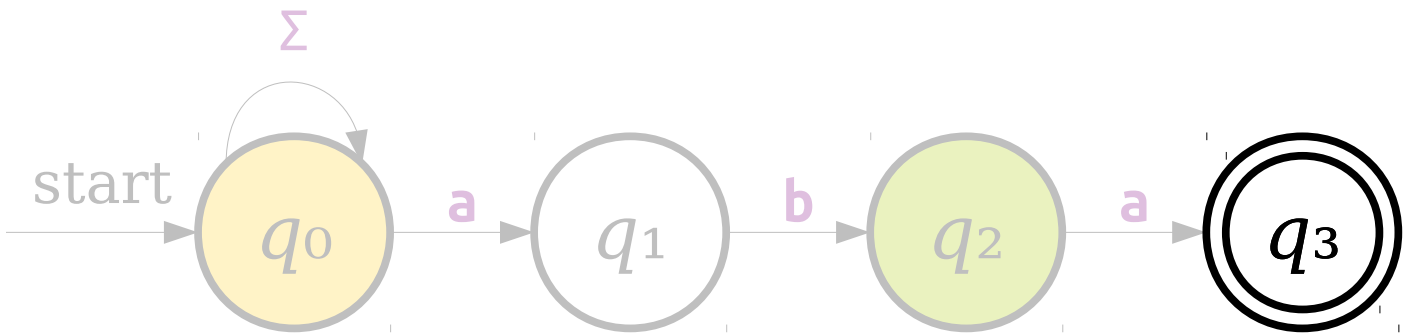



| | | | |
|---|---|---|---|
| a | b | a | b |

# Massive Parallelism




We're not in any accepting state, so no possible path accepts.

| | | | |
|---|---|---|---|
| a | b | a | b |

# Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- (Here's a rigorous explanation about how this works; read this on your own time).
  - Start off in the set of all states formed by taking the start state and including each state that can be reached by zero or more ε-transitions.
  - When you read a symbol `a` in a set of states *S*:
    - Form the set *S'* of states that can be reached by following a single `a` transition from some state in *S*.
    - Your new set of states is the set of states in *S'*, plus the states reachable from *S'* by following zero or more ε-transitions.

# Designing NFAs

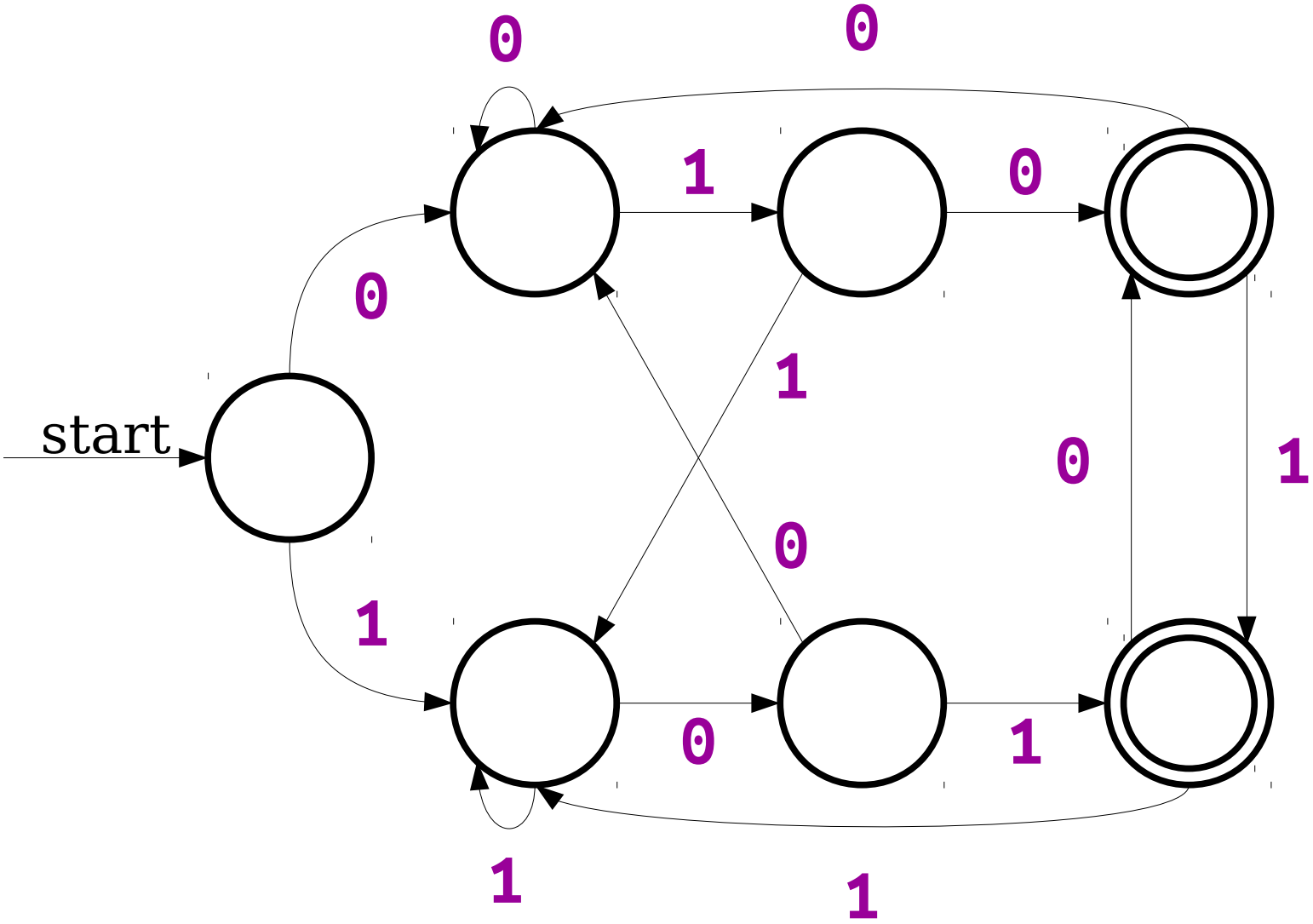
- ***Embrace the nondeterminism!***
- Good model: ***Guess-and-check***:
  - Is there some information that you'd really like to have? Have the machine *nondeterministically guess* that information.
  - Then, have the machine *deterministically check* that the choice was correct.
- The *guess* phase corresponds to trying lots of different options.
- The *check* phase corresponds to filtering out bad guesses or wrong options.

# Guess-and-Check

$L = \{ w \in \{\texttt{0}, \texttt{1}\}^* \mid w$ ends in $\texttt{010}$ or $\texttt{101} \}$

# Guess-and-Check

$L = \{ w \in \{\mathtt{0}, \mathtt{1}\}^* \mid w \text{ ends in } \mathtt{010} \text{ or } \mathtt{101} \}$



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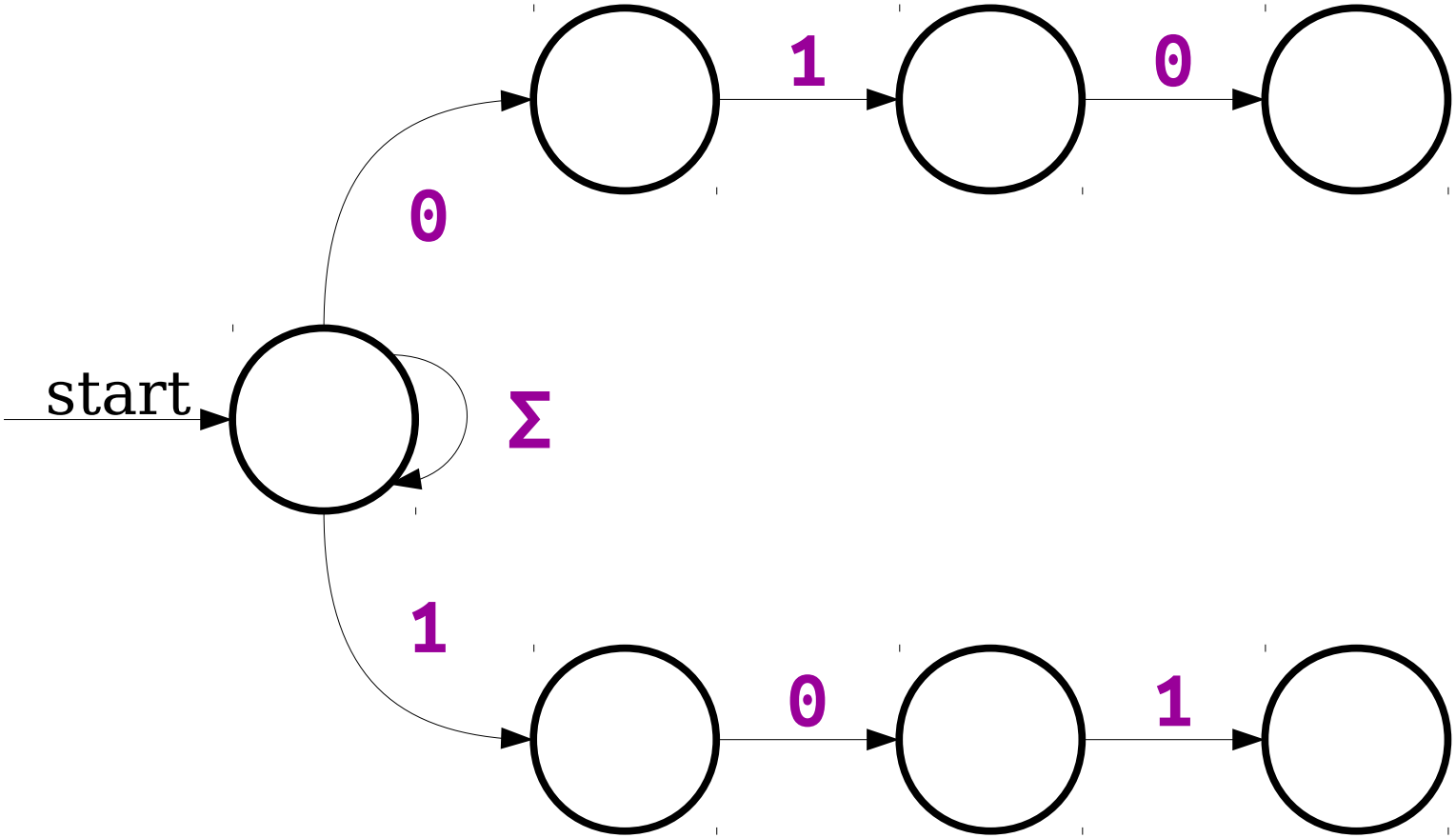

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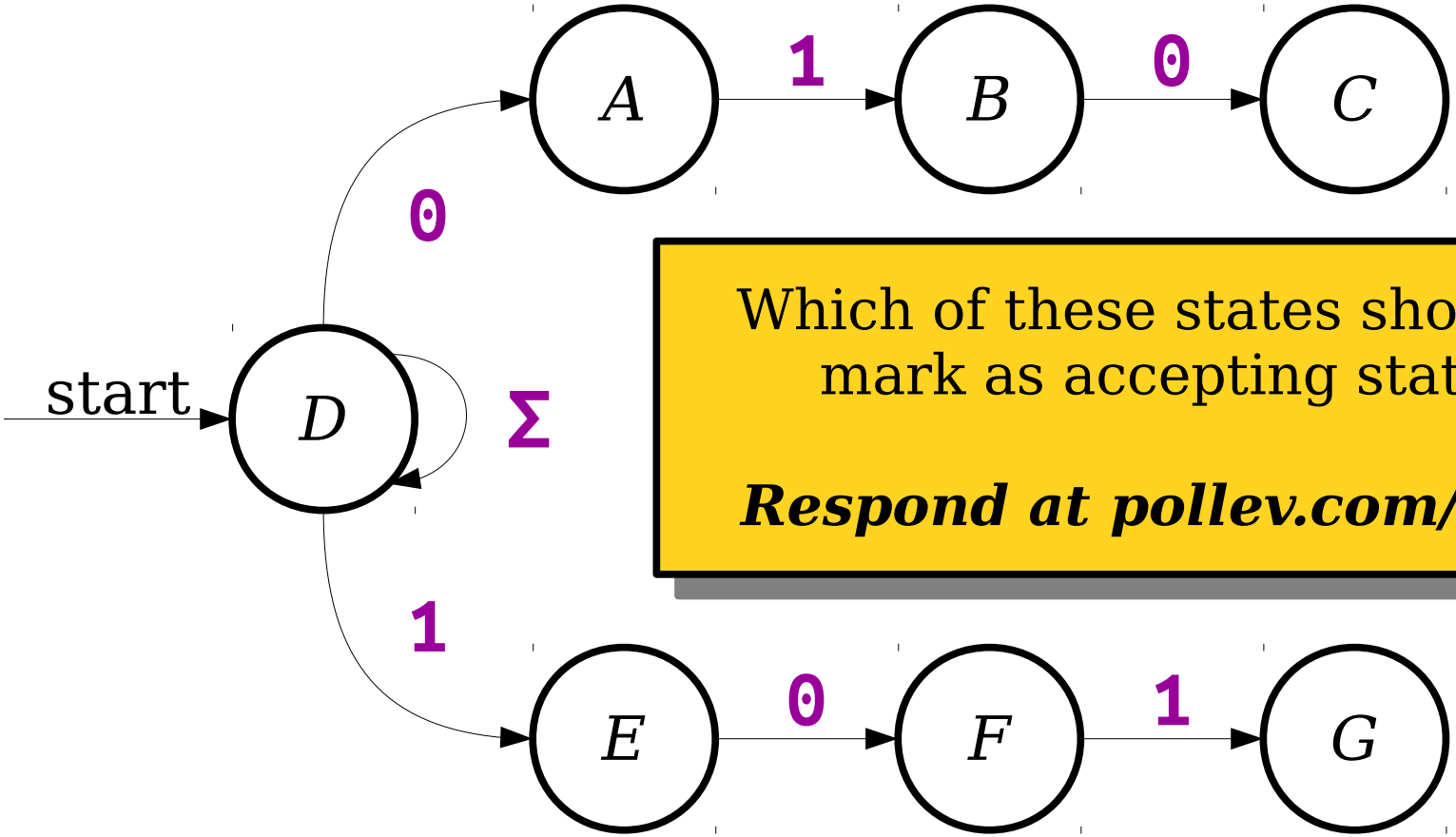
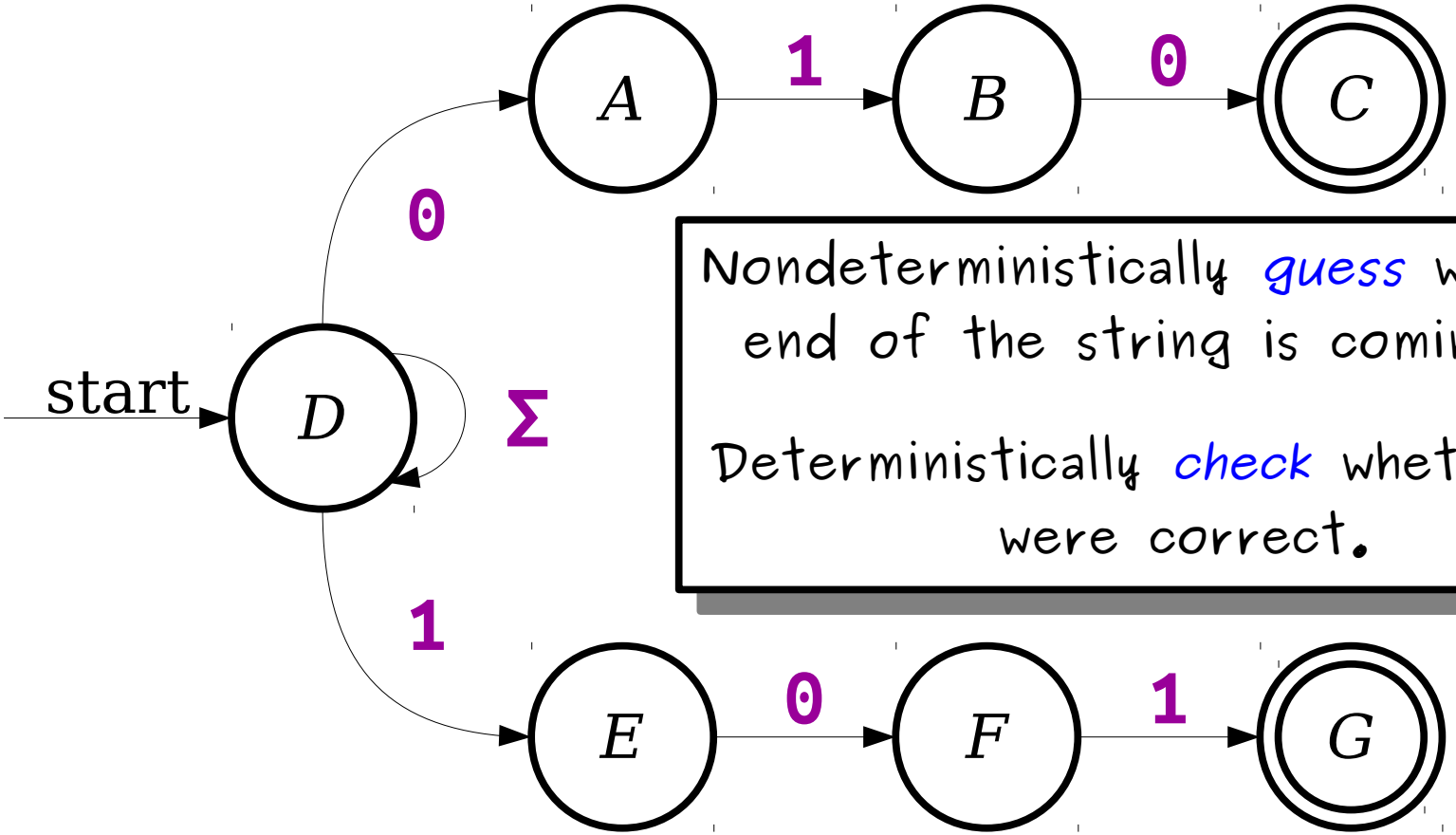



Which of these states should we mark as accepting states?

***Respond at pollev.com/cs103***

# Guess-and-Check

$L = \{ w \in \{\mathtt{0}, \mathtt{1}\}^* \mid w \text{ ends in } \mathtt{010} \text{ or } \mathtt{101} \}$

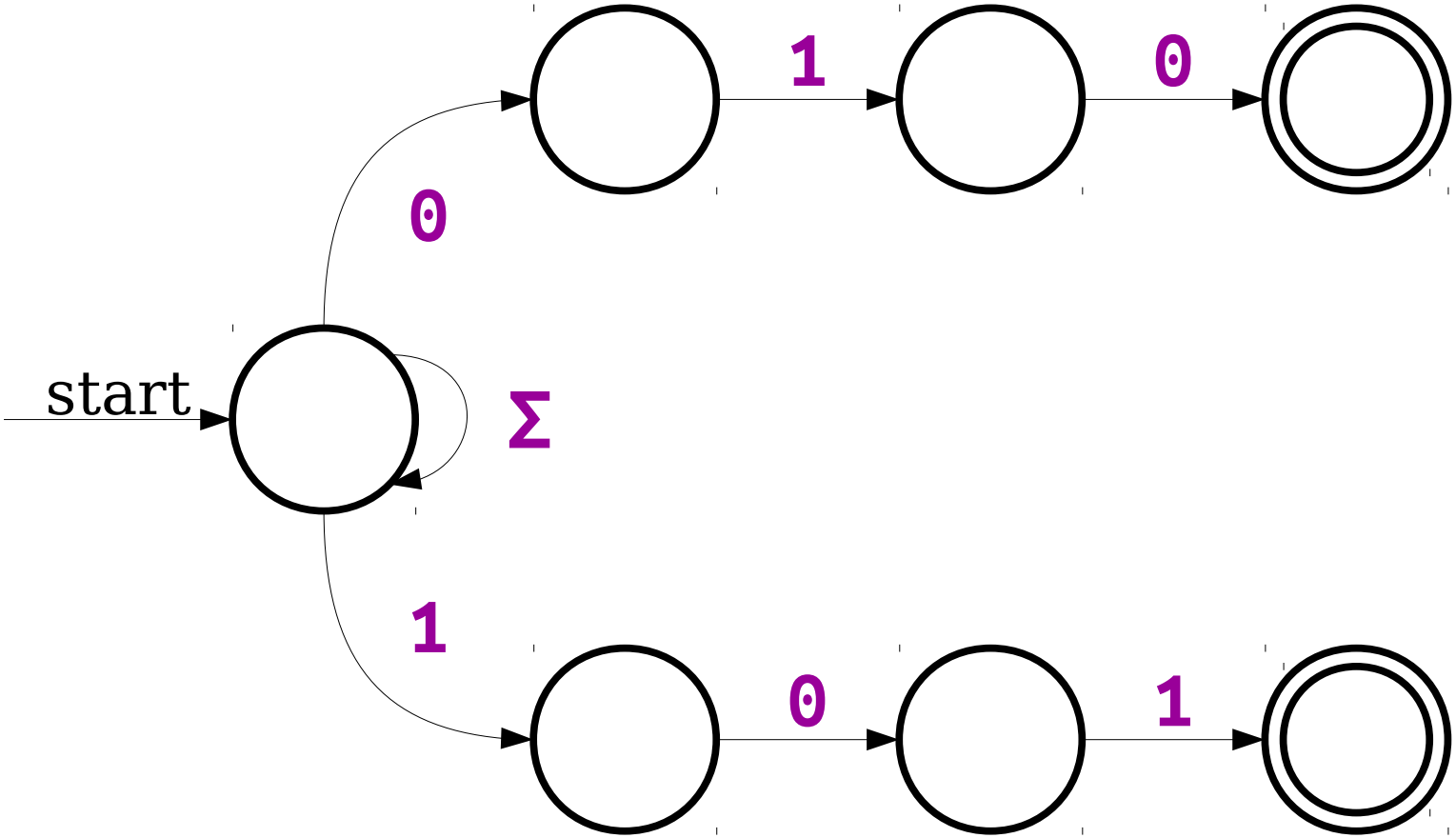



Nondeterministically *guess* when the end of the string is coming up.

Deterministically *check* whether you were correct.

# Guess-and-Check

$L = \{ w \in \{\mathtt{0}, \mathtt{1}\}^* \mid w \text{ ends in } \mathtt{010} \text{ or } \mathtt{101} \}$



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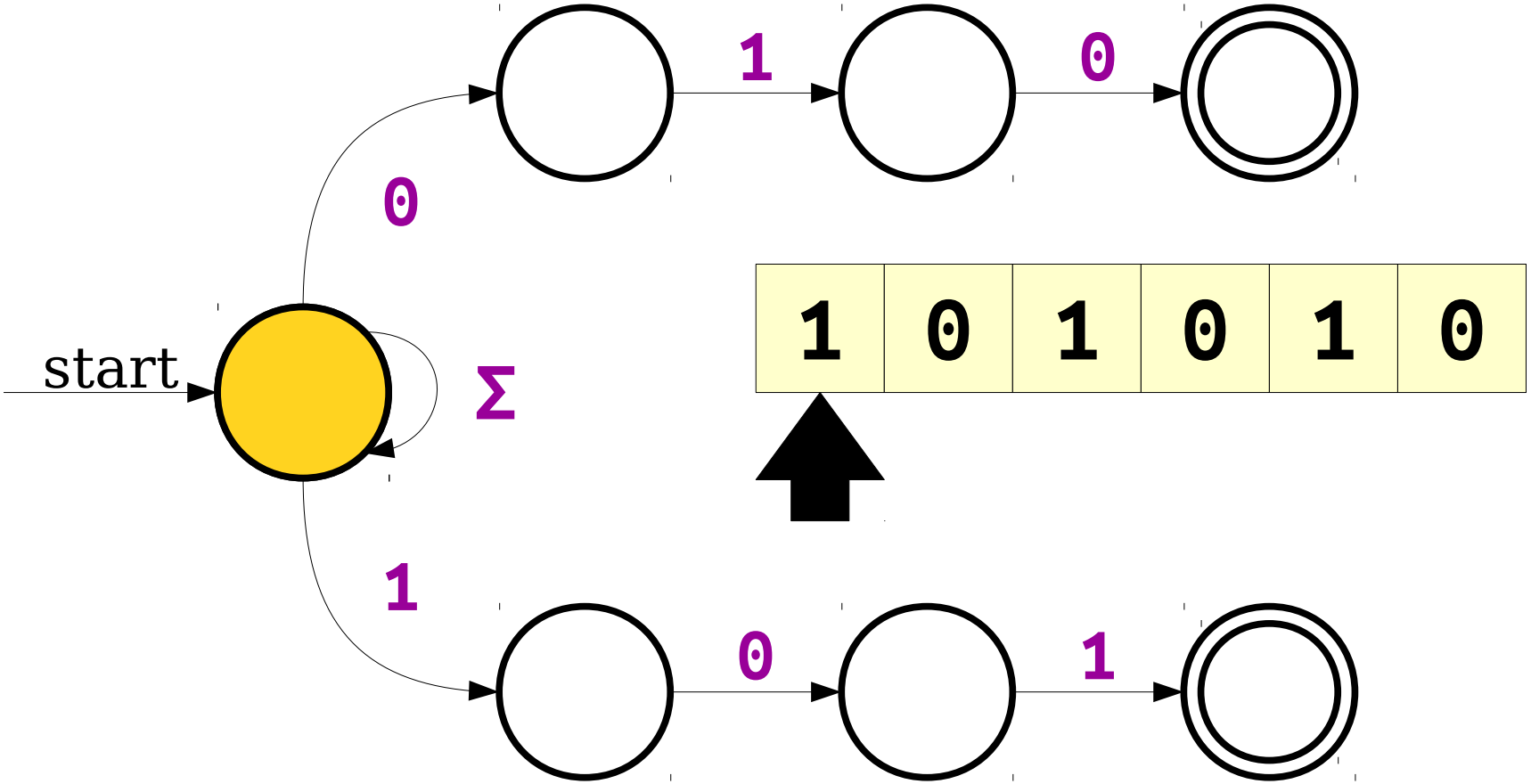

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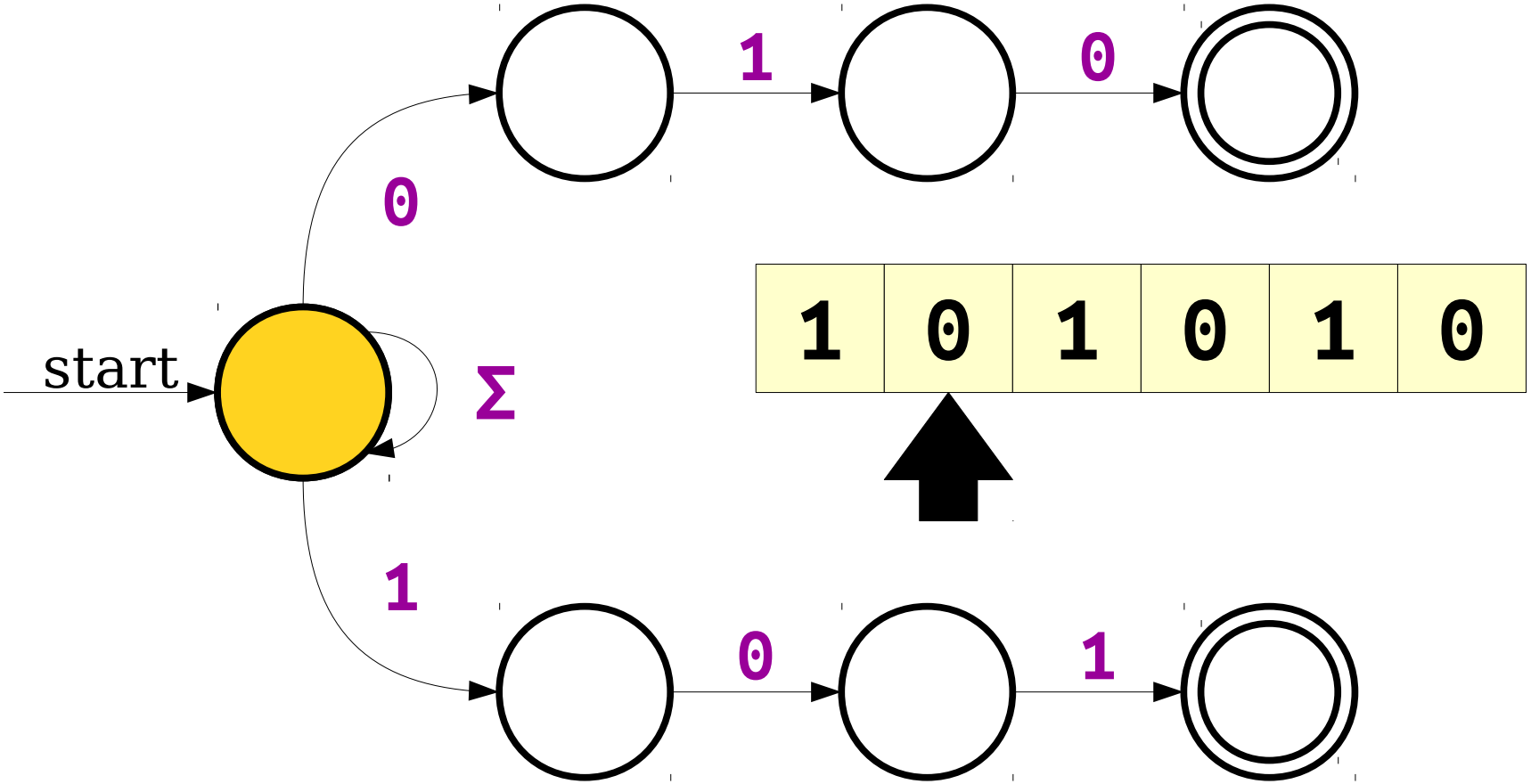

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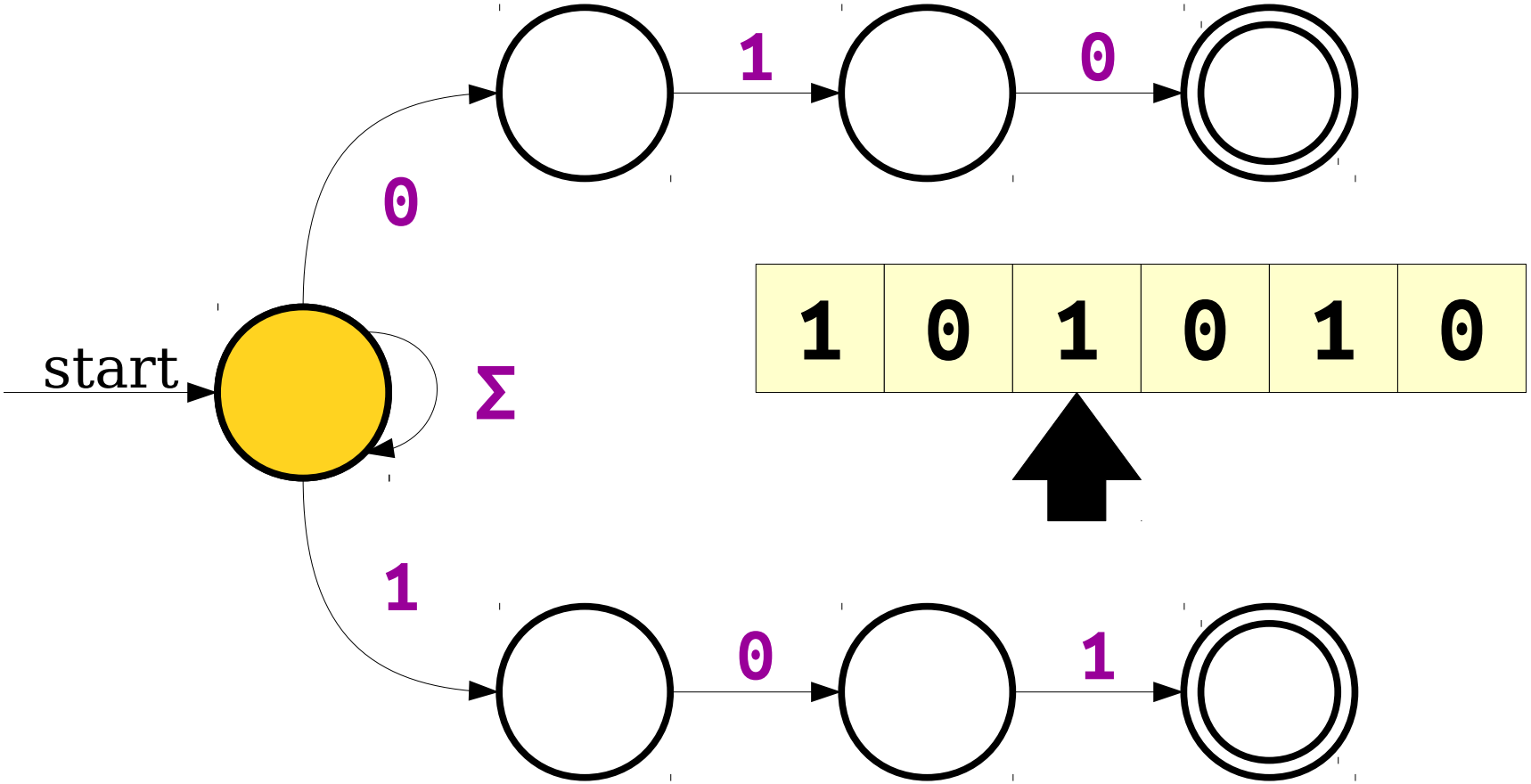

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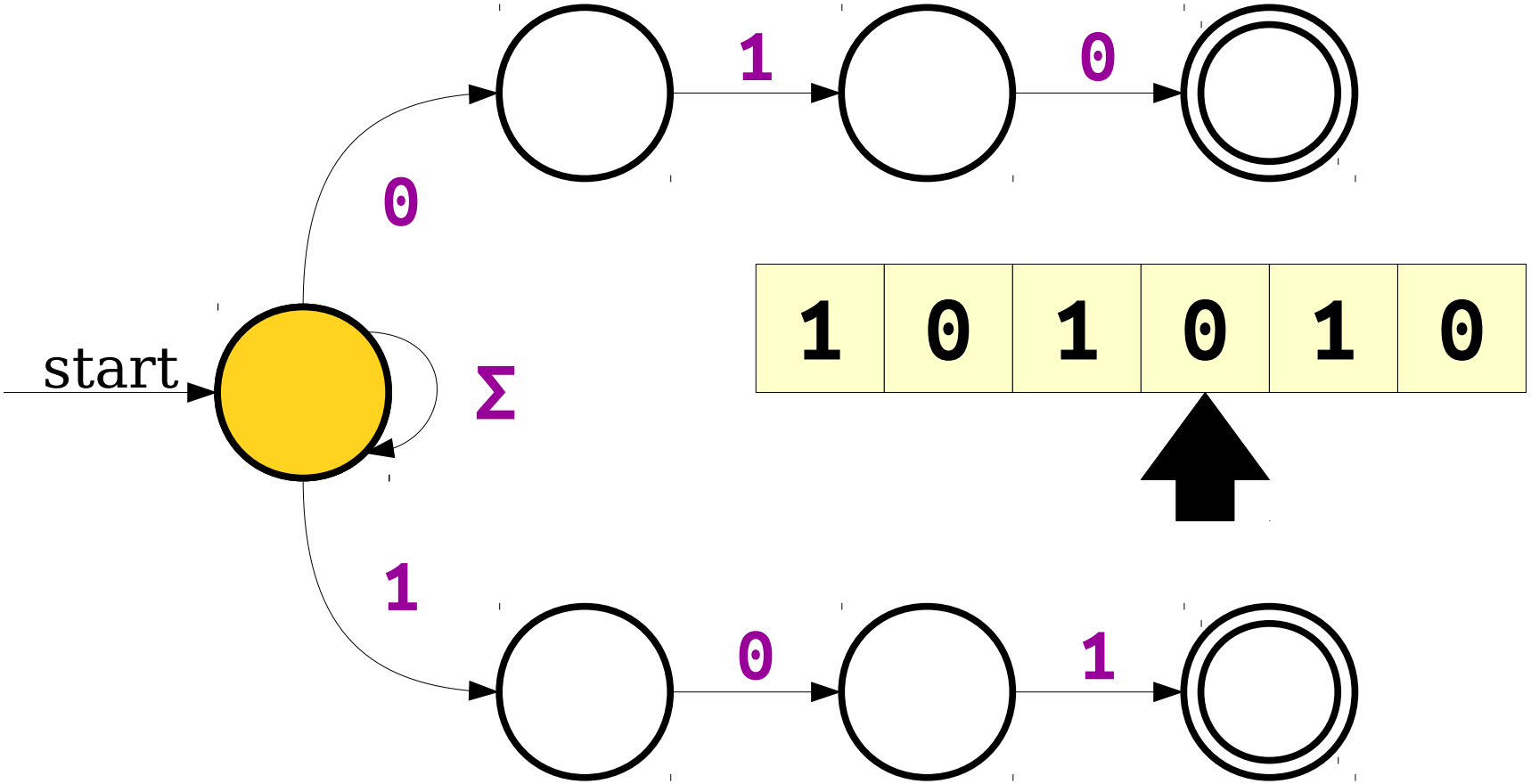

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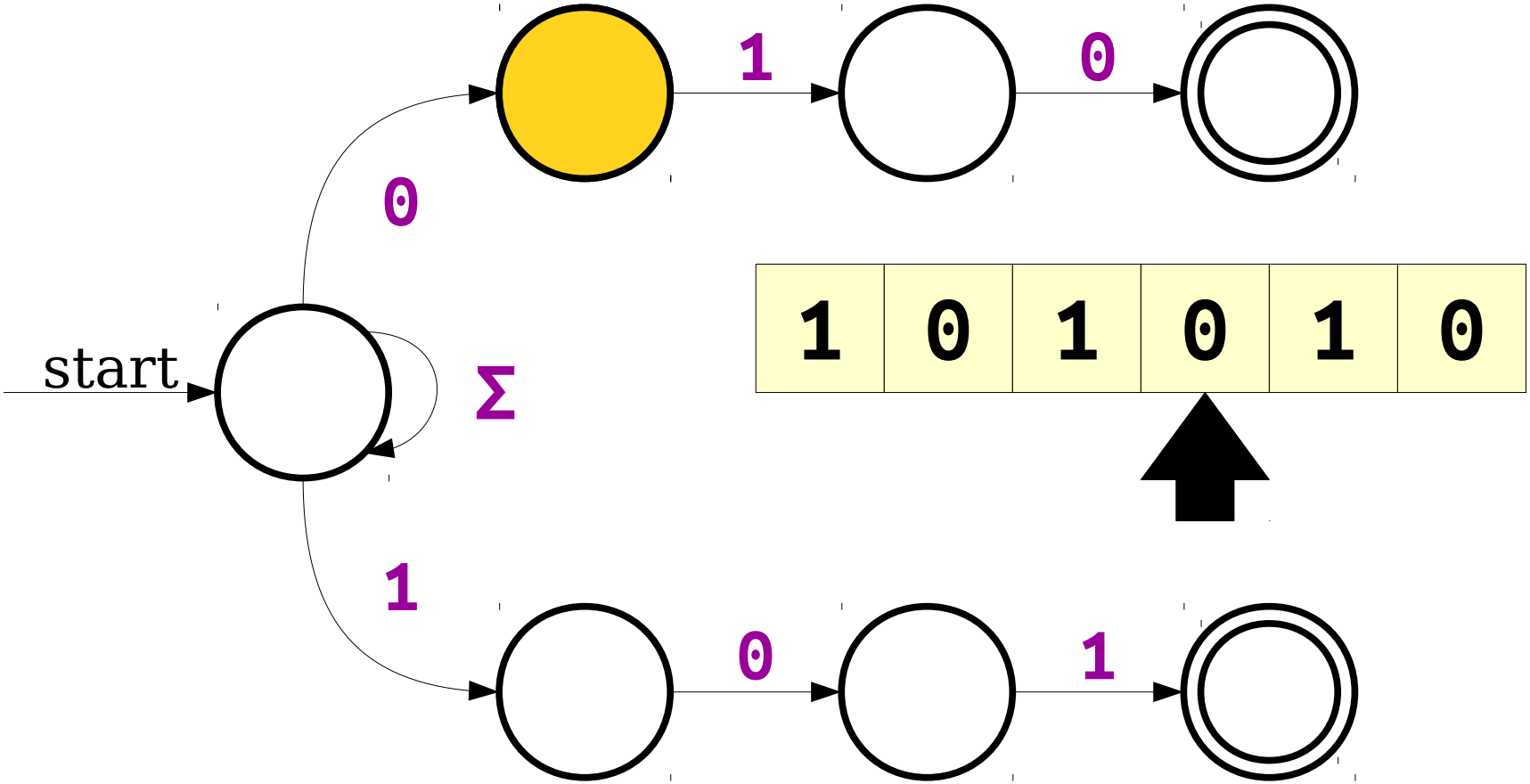

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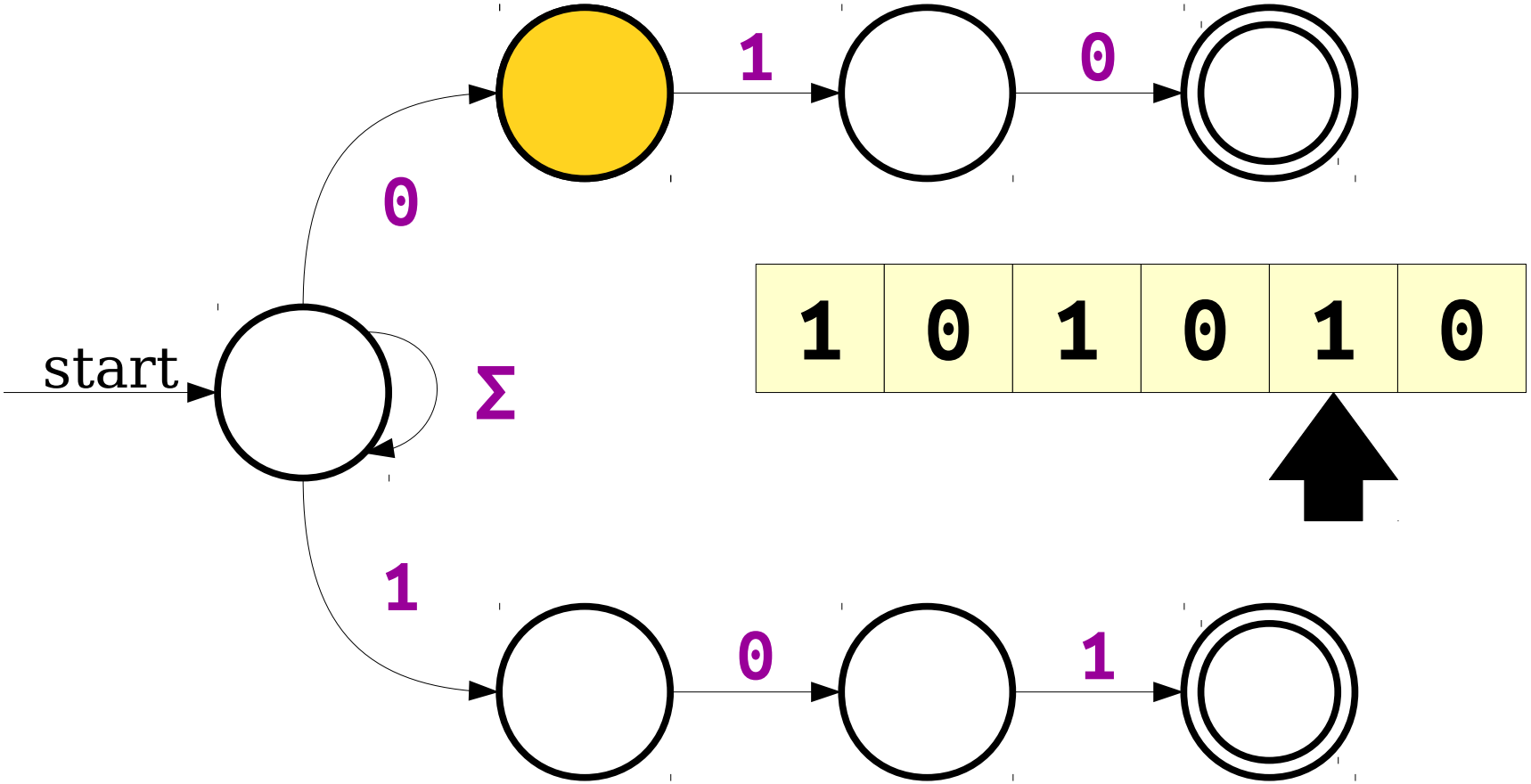

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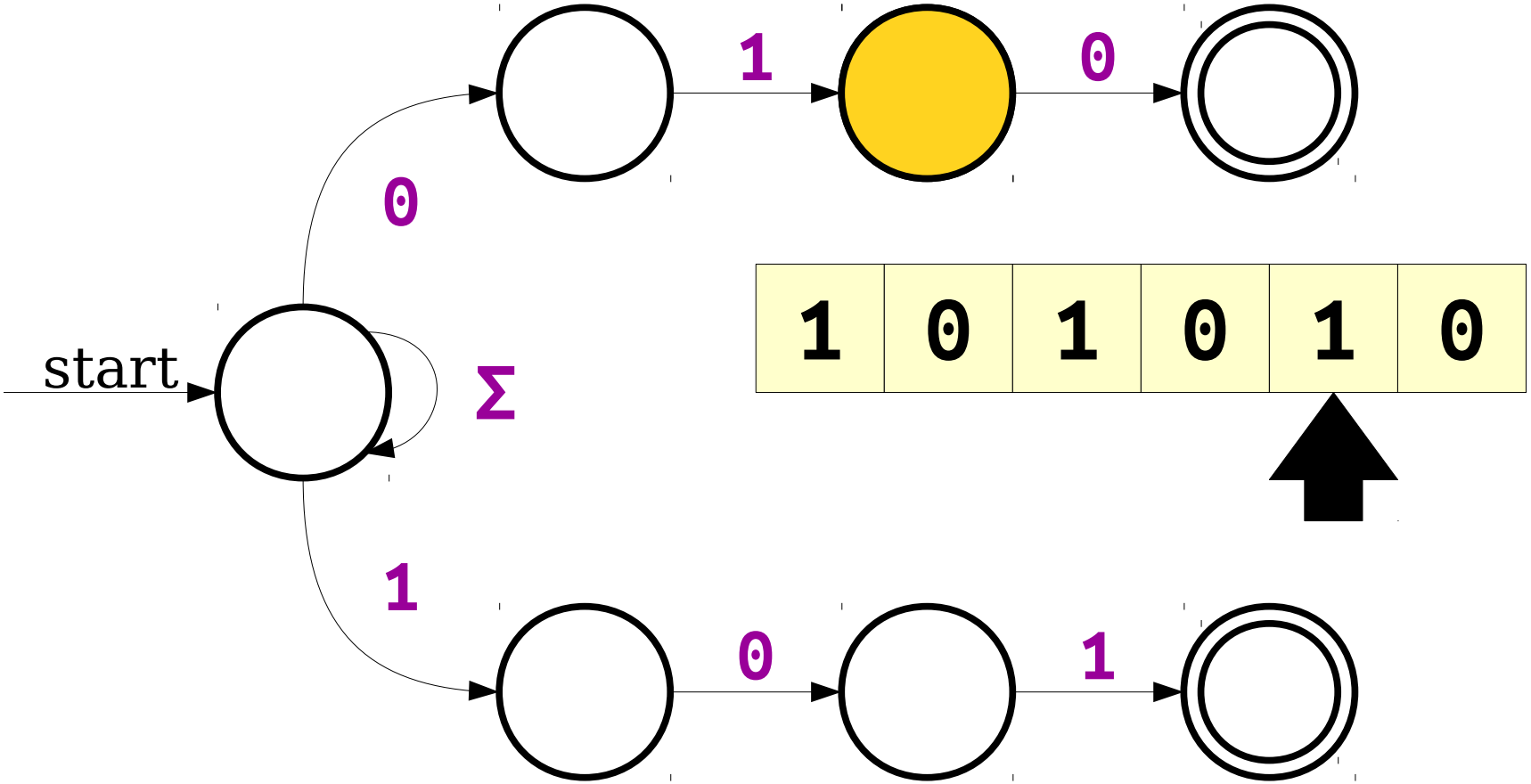

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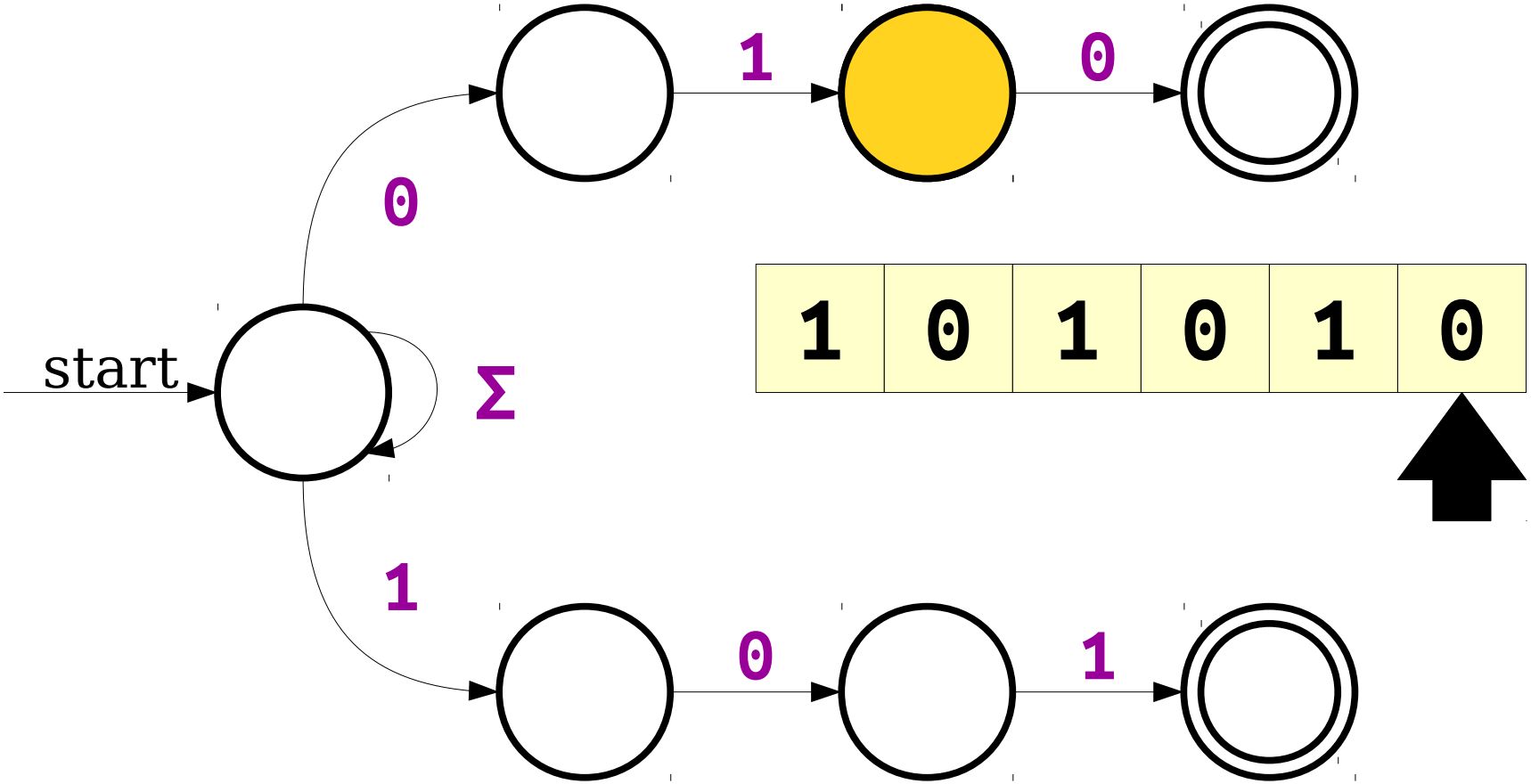

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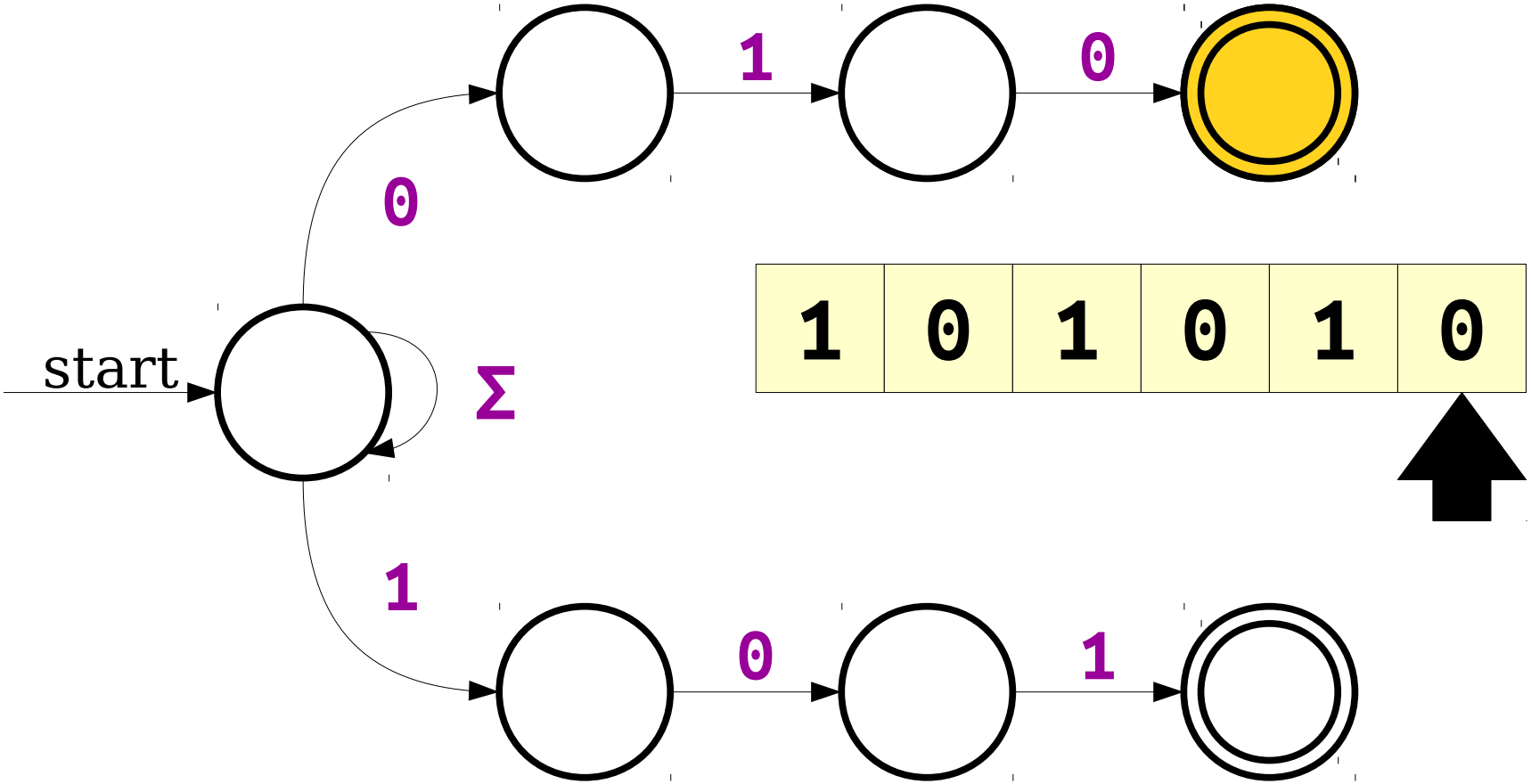

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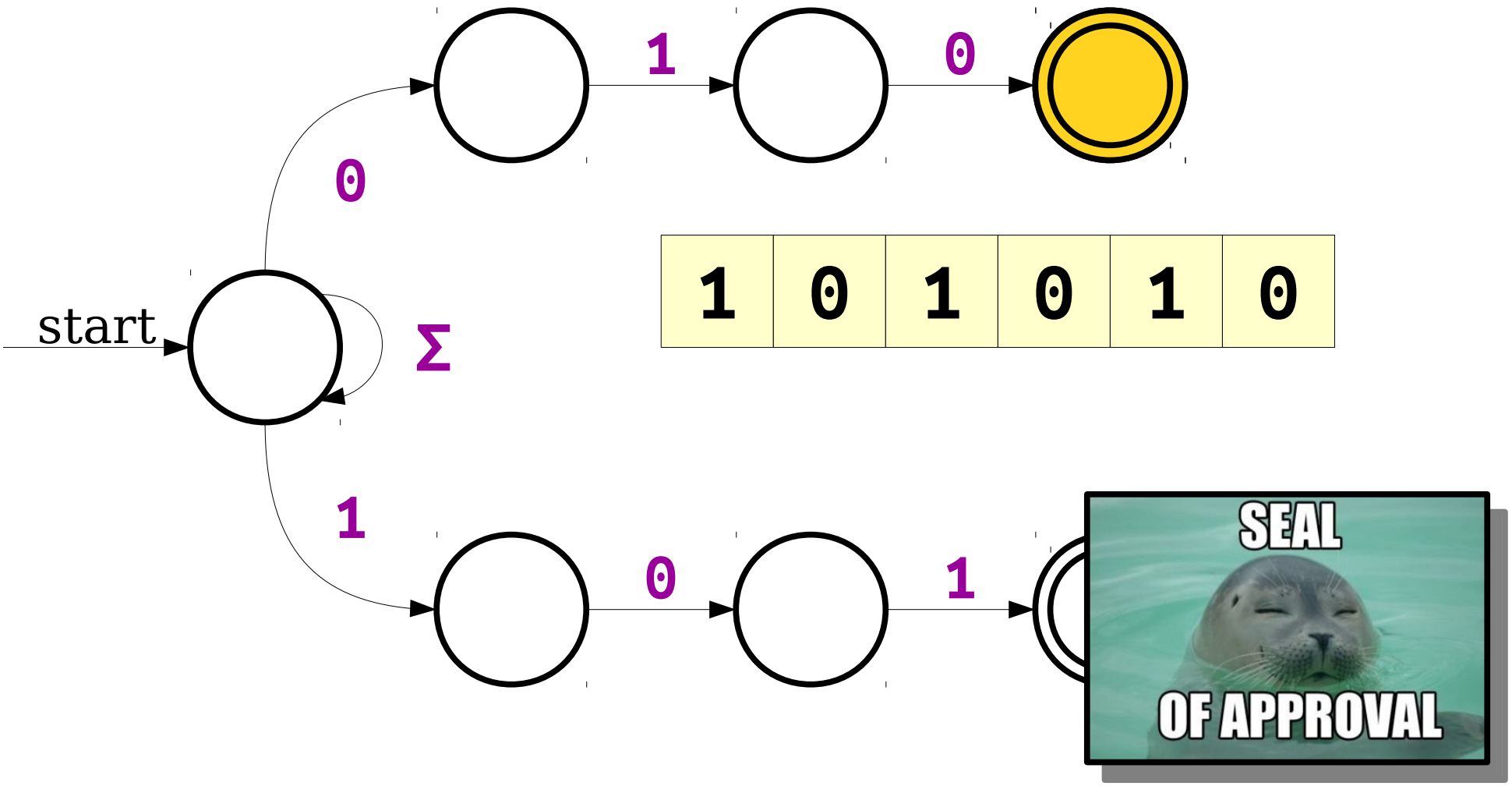
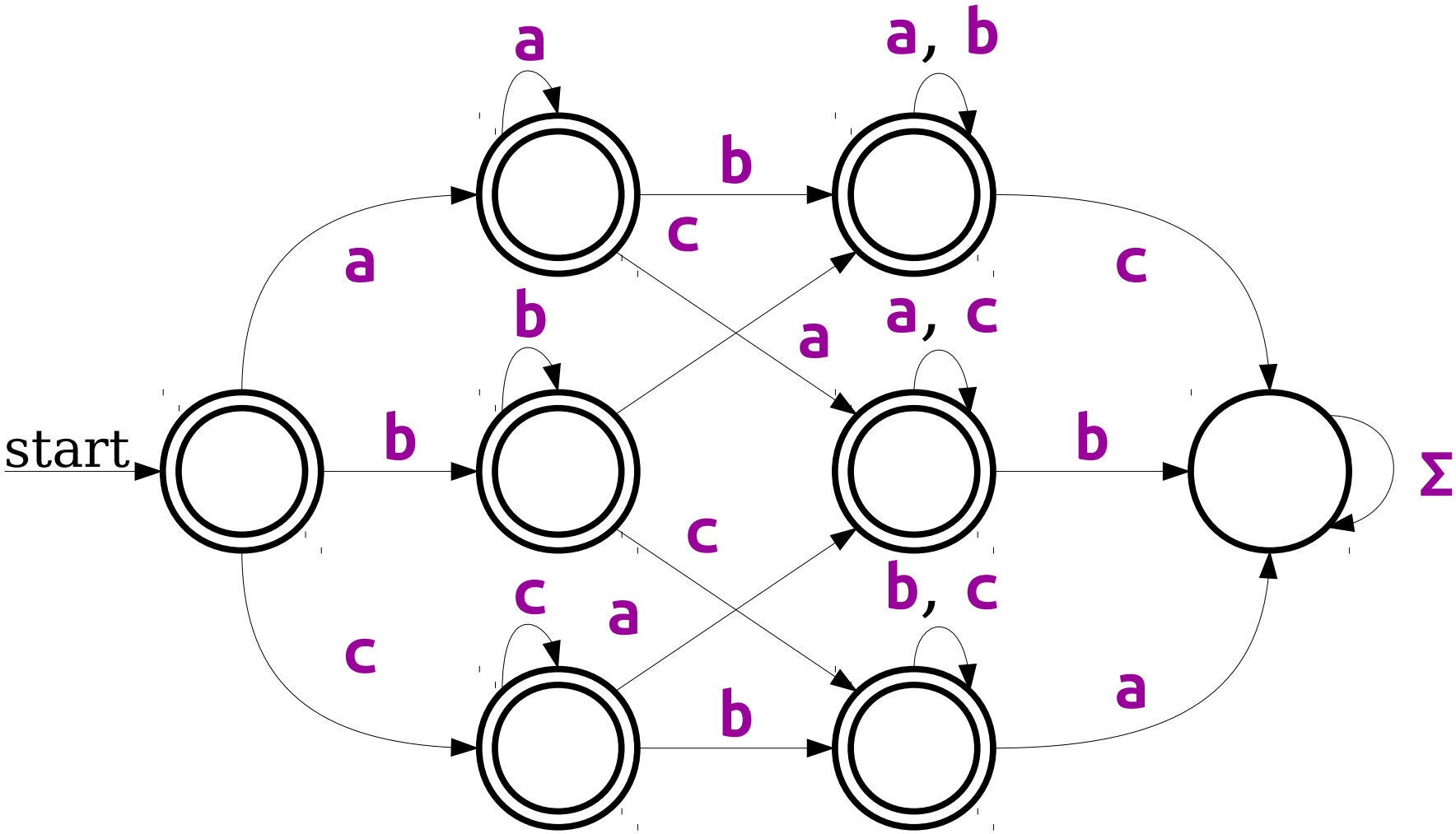


# Guess-and-Check

$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of $\mathtt{a}$, $\mathtt{b}$, or $\mathtt{c}$ is not in $w \}$

# Guess-and-Check

$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$



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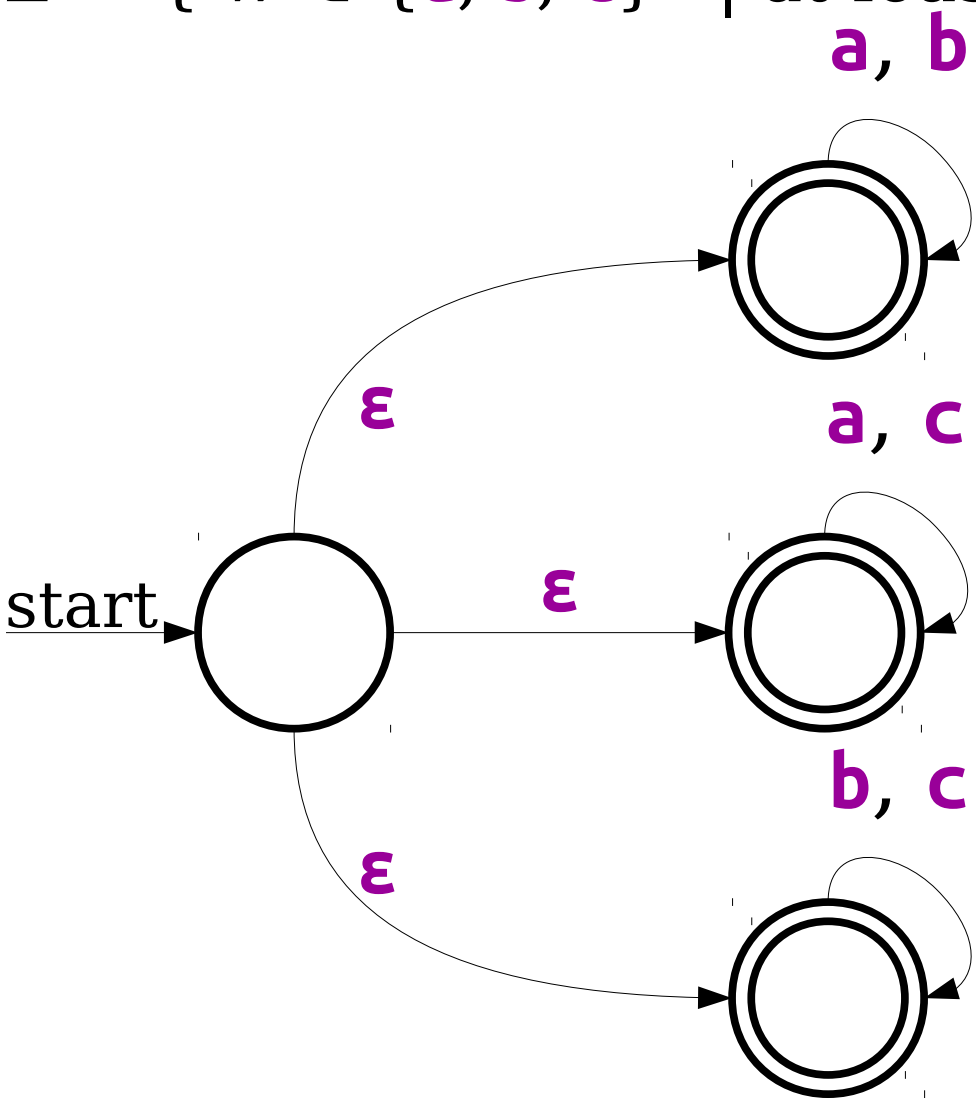
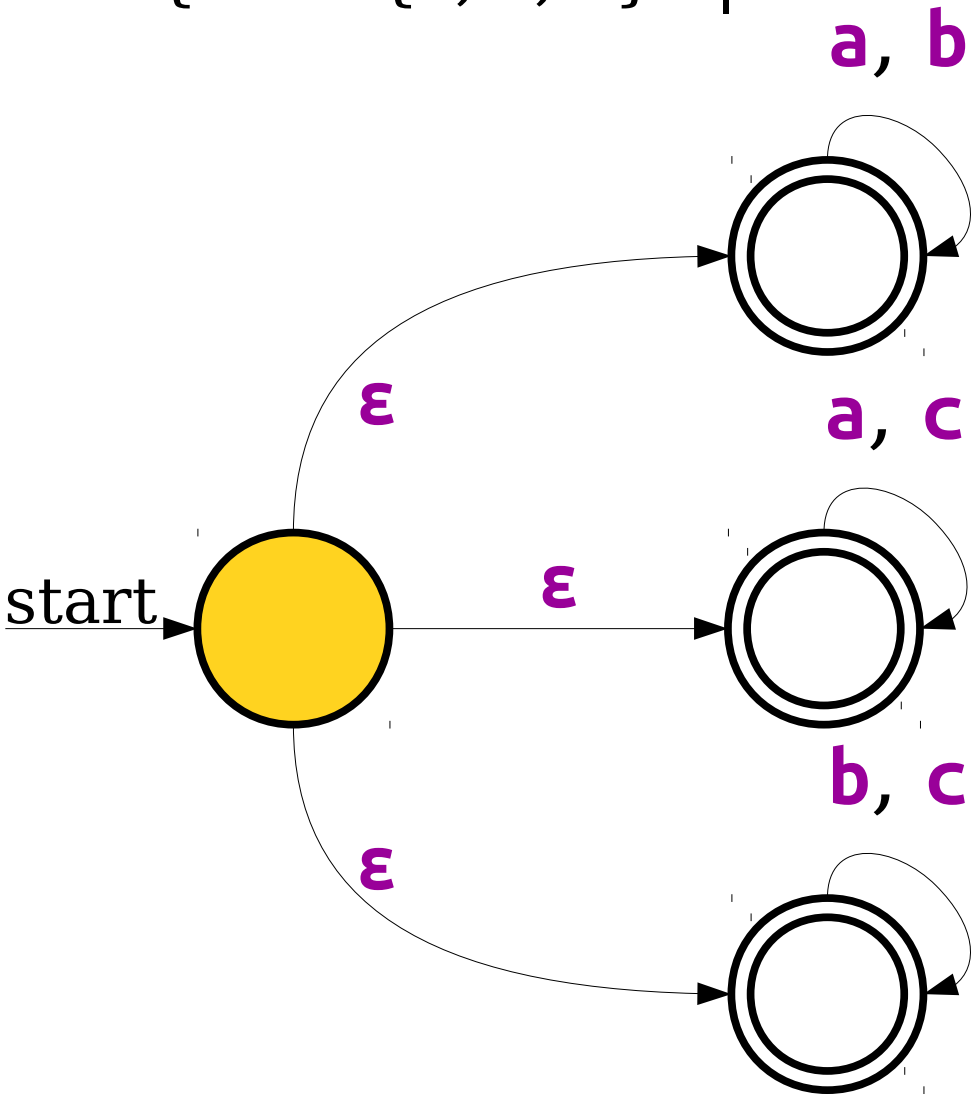


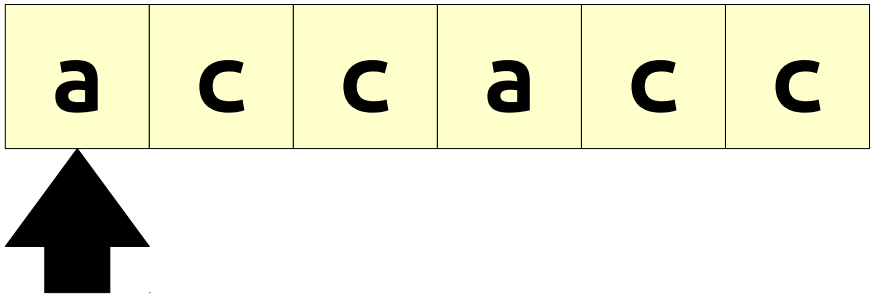

Nondeterministically *guess* which character is missing.

Deterministically *check* whether that character is indeed missing.

# Guess-and-Check

$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$



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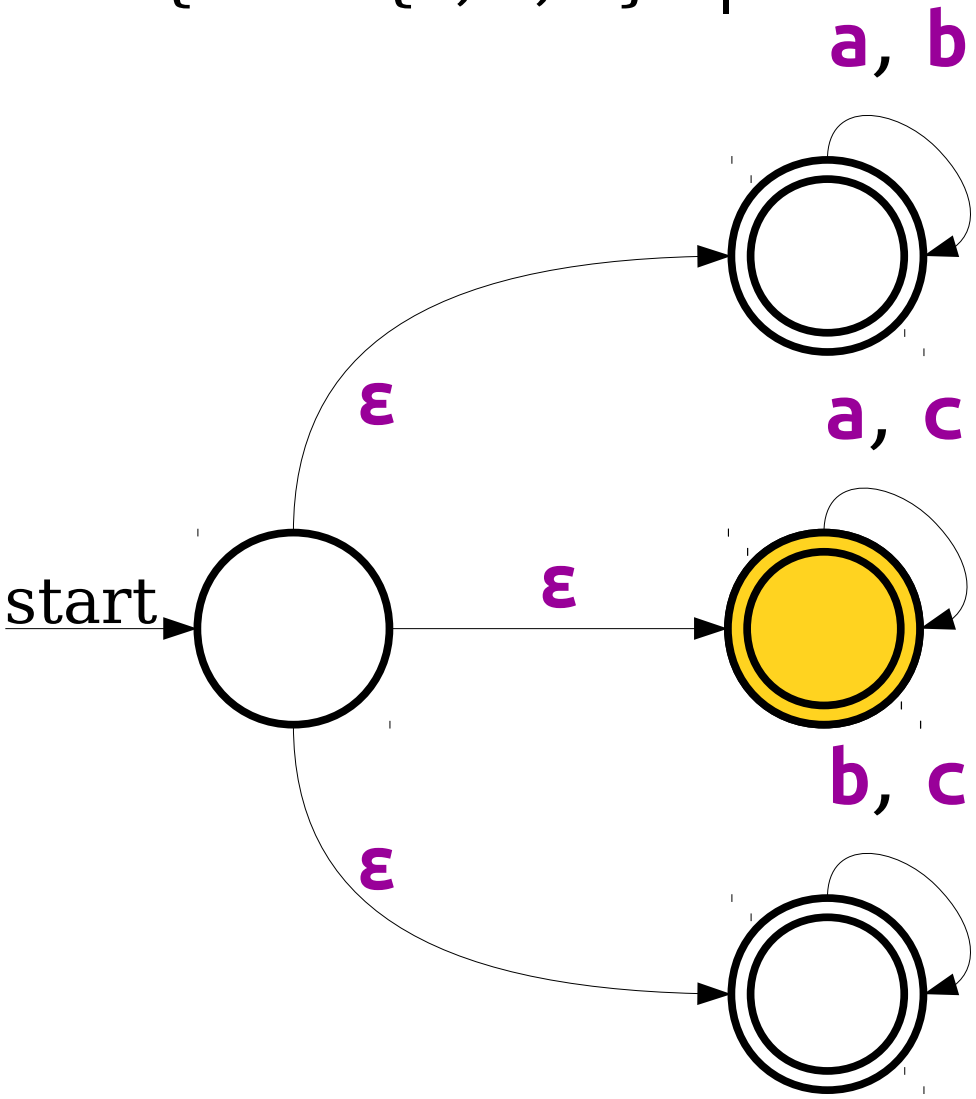


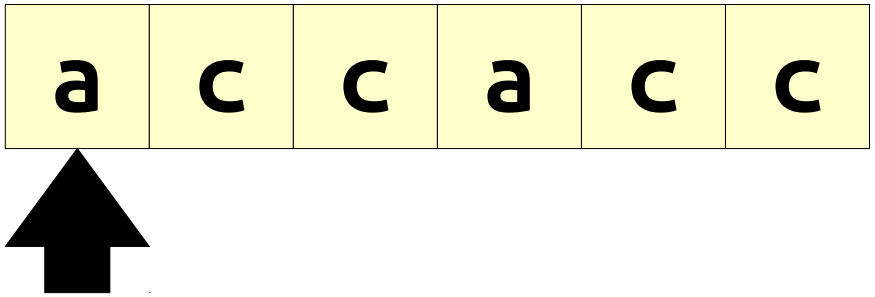

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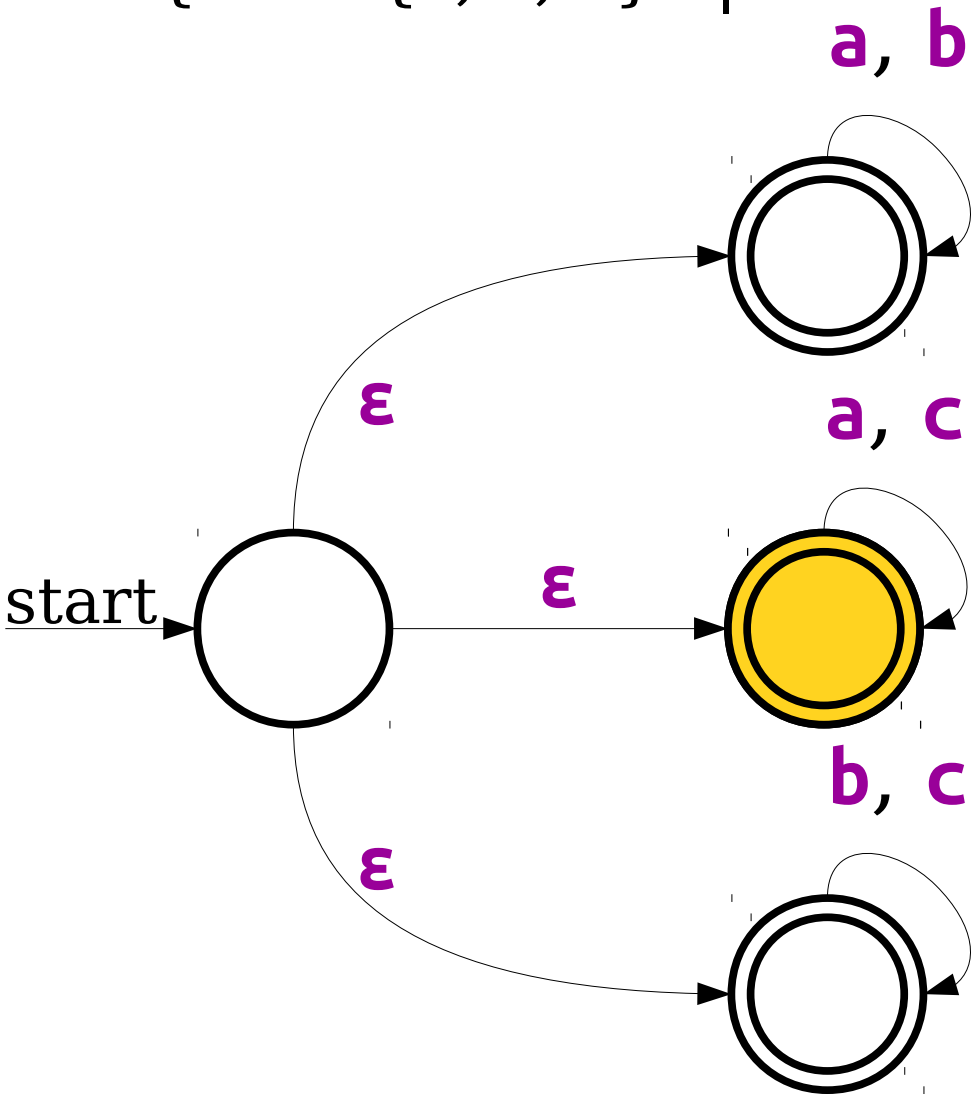
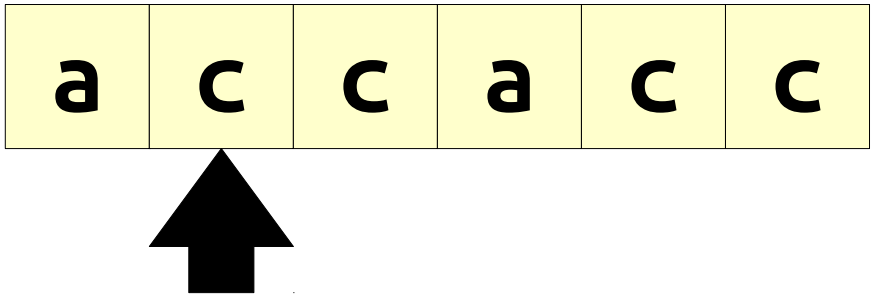
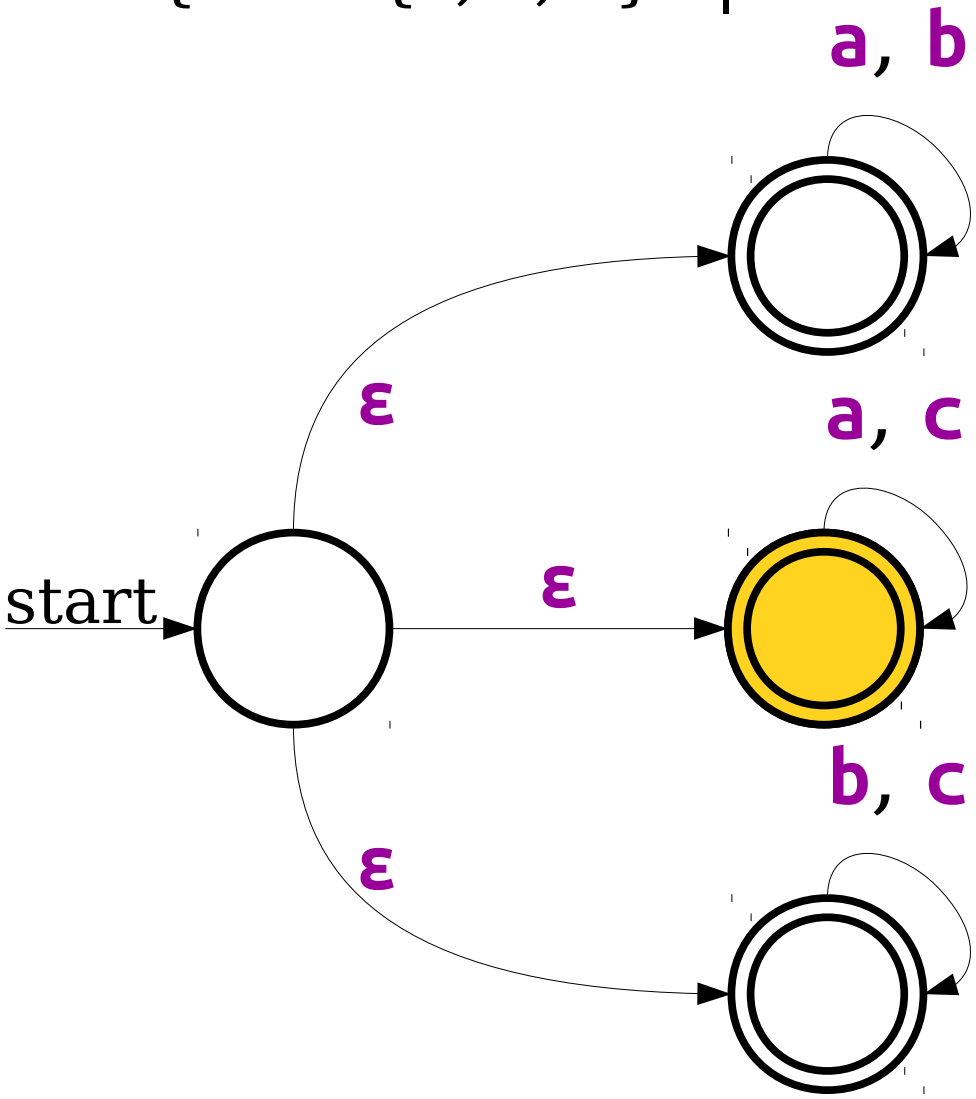
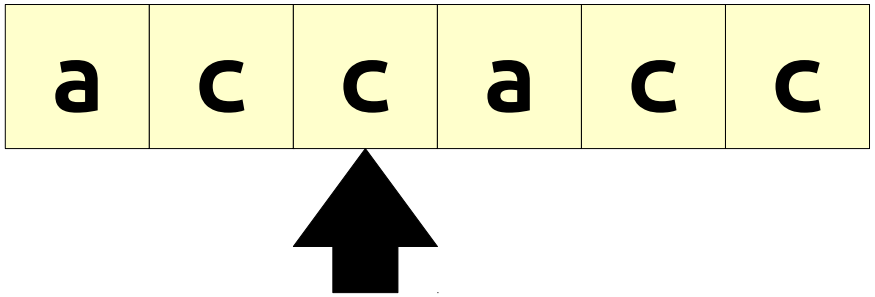


# Guess-and-Check

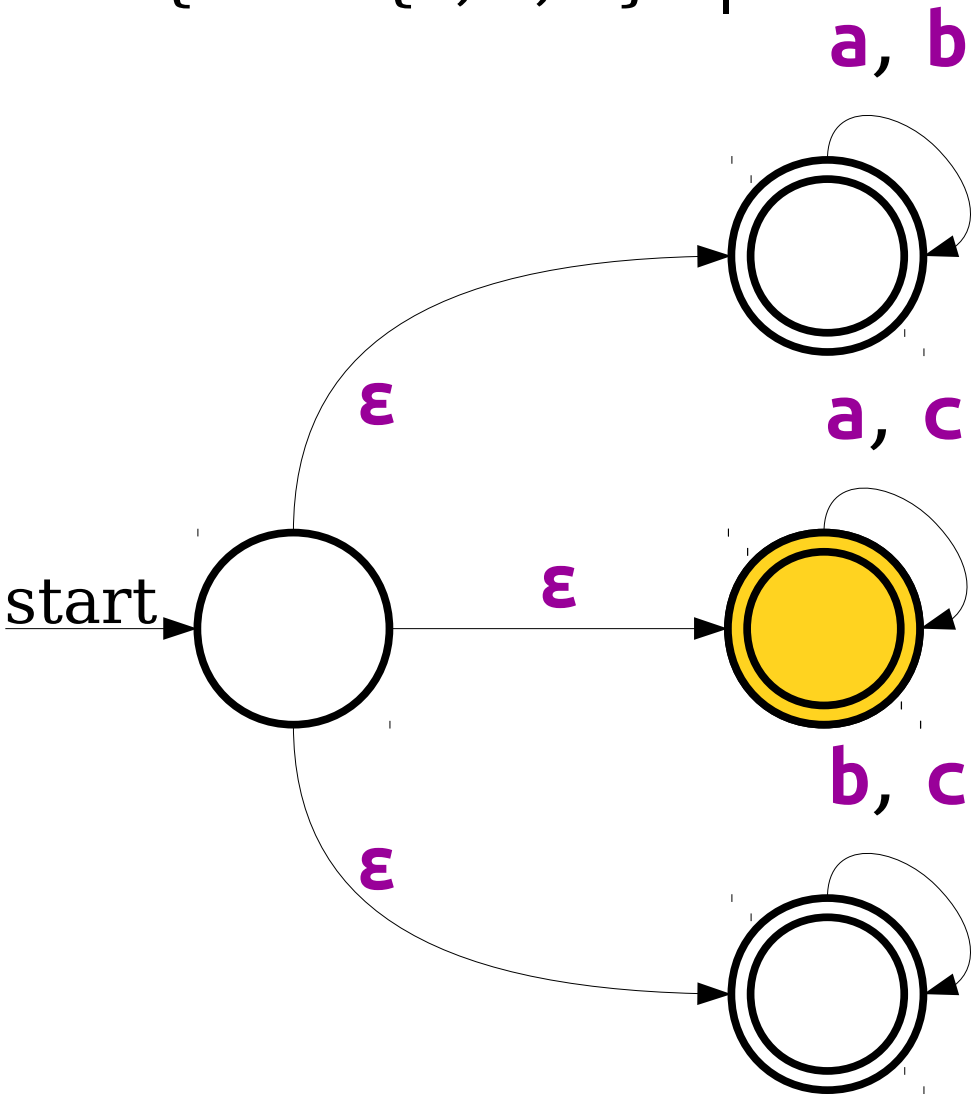
$L = \{ w \in \{a, b, c\}^* \mid$ at least one of a, b, or c is not in $w \}$

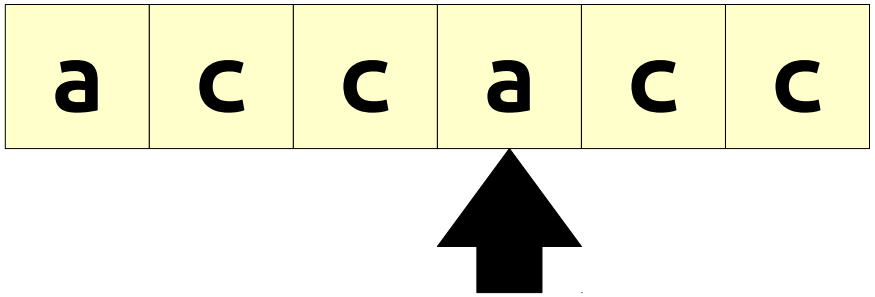


# Guess-and-Check

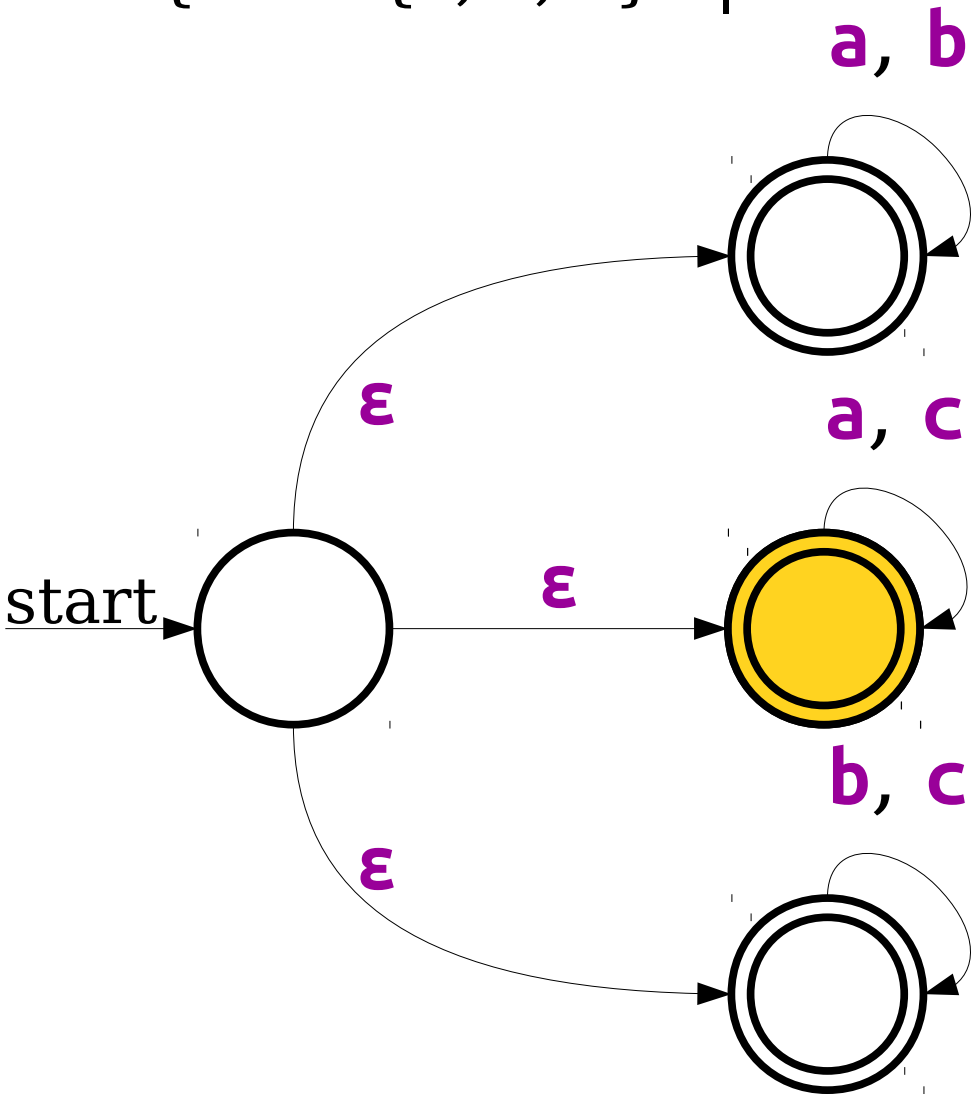
$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$

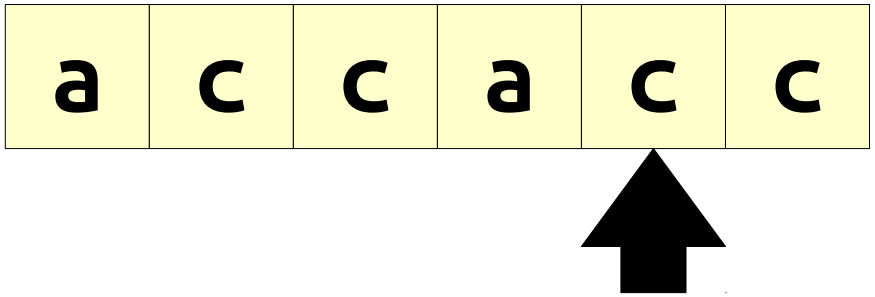


# Guess-and-Check

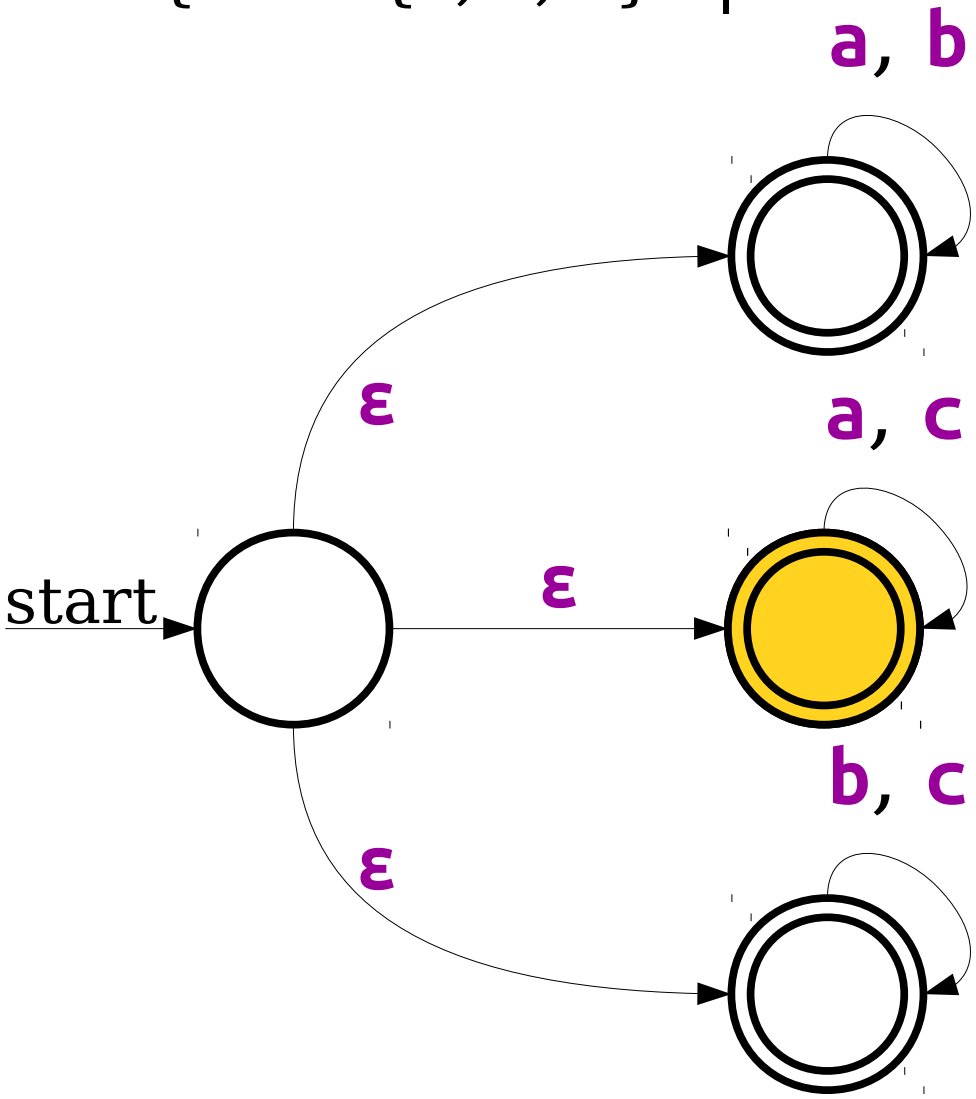
$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$

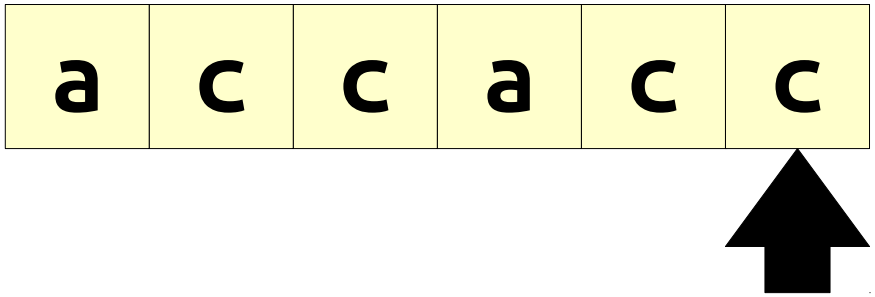


# Guess-and-Check

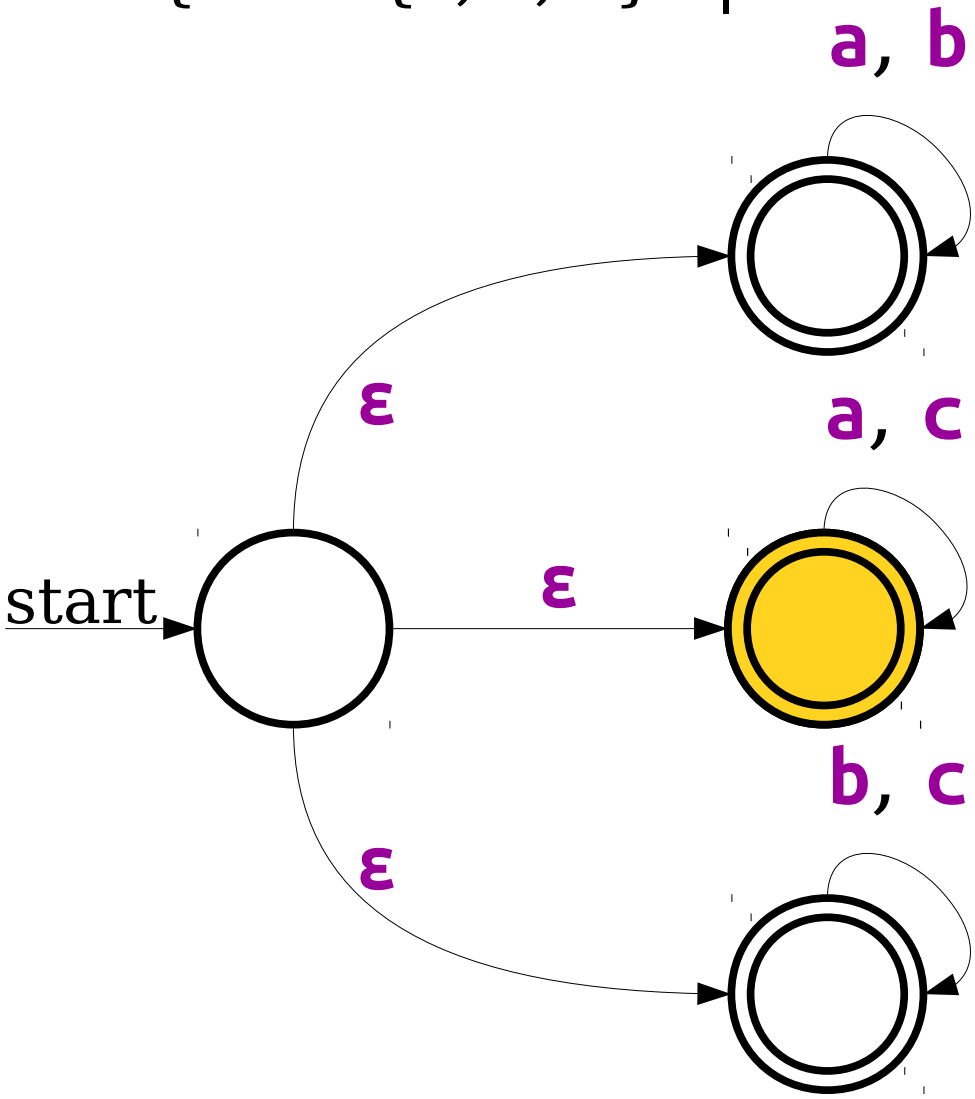
$L = \{ w \in \{\texttt{a}, \texttt{b}, \texttt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$



# Guess-and-Check

$L = \{ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$

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| a | c | c | a | c | c |

Time-Out For Announcements!

# Midterm Exam on Friday!

- Our midterm exam will be on Friday, July 28th from 4:30 – 7:30 PM in Shriram 104 (our normal lecture room).
- You're responsible for lectures up to the end of week 3 and topics from PS1 – PS3. Later lectures and problem sets won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is open-book, open-note, and closed-other-humans/AI.

Back to CS103!

# Just how powerful are NFAs?

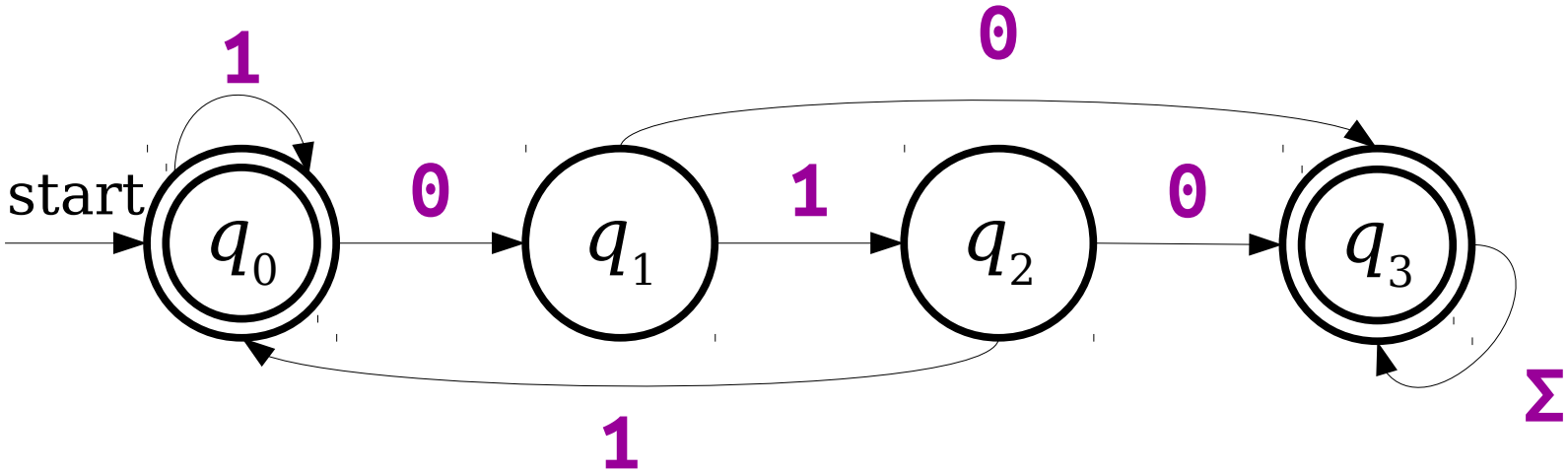
# NFAs and DFAs

- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

***Thought Experiment:***

How would you simulate a finite automata in software?

# Tabular DFAs

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| $q_2$ | | |
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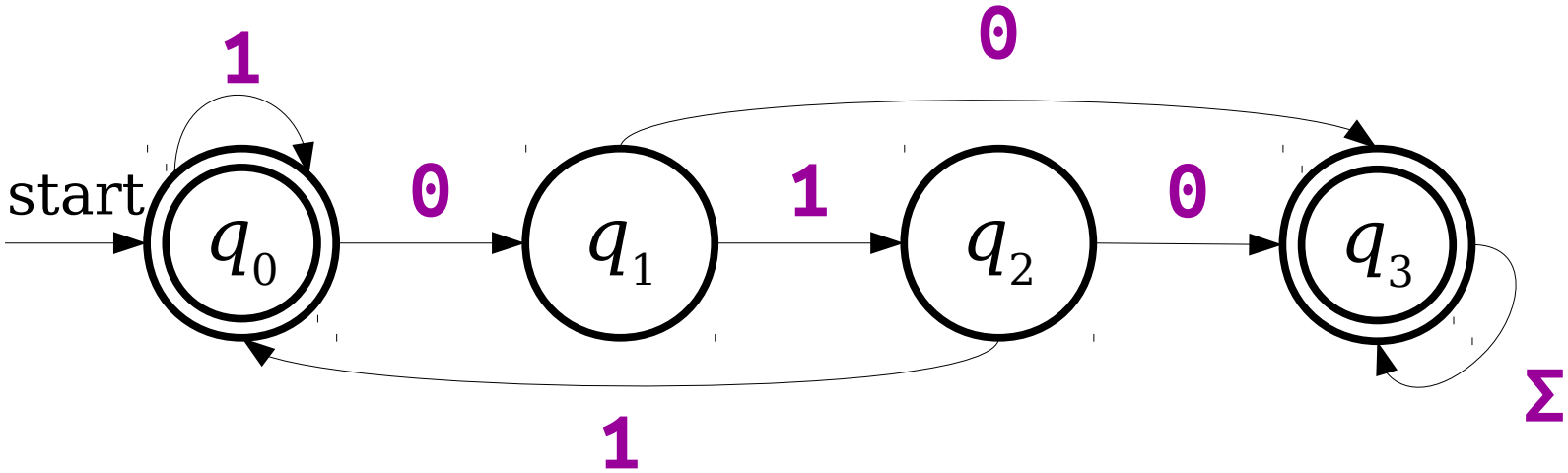
# Tabular DFAs




| | 0 | 1 |
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| $q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_3$ | $q_2$ |
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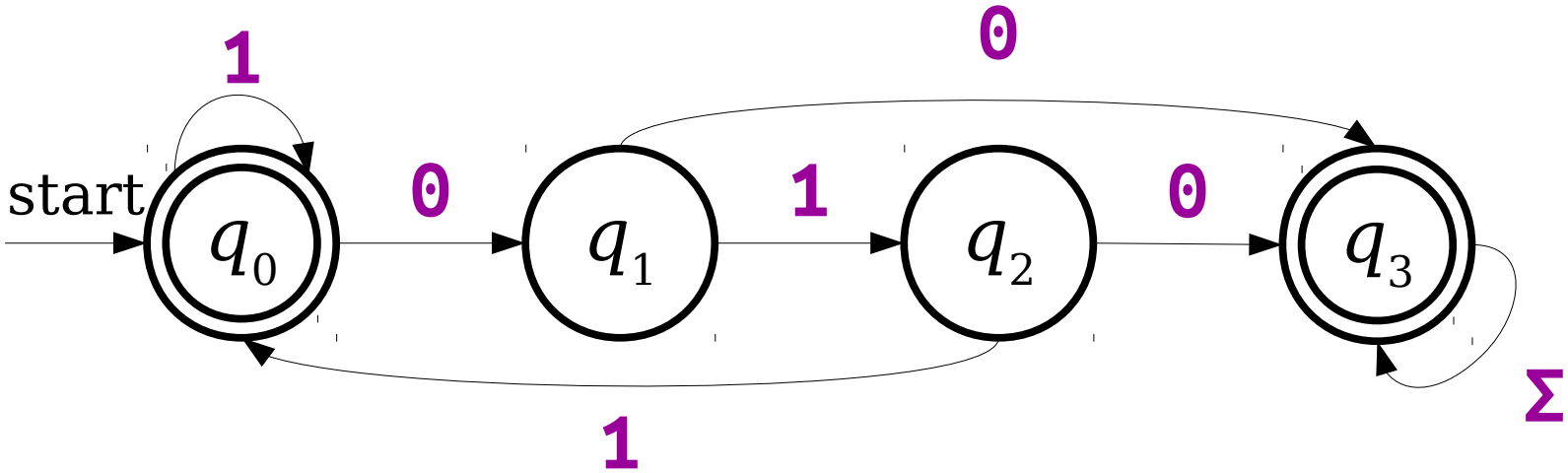
| $q_3$ | $q_3$ | $q_3$ |

# Tabular DFAs

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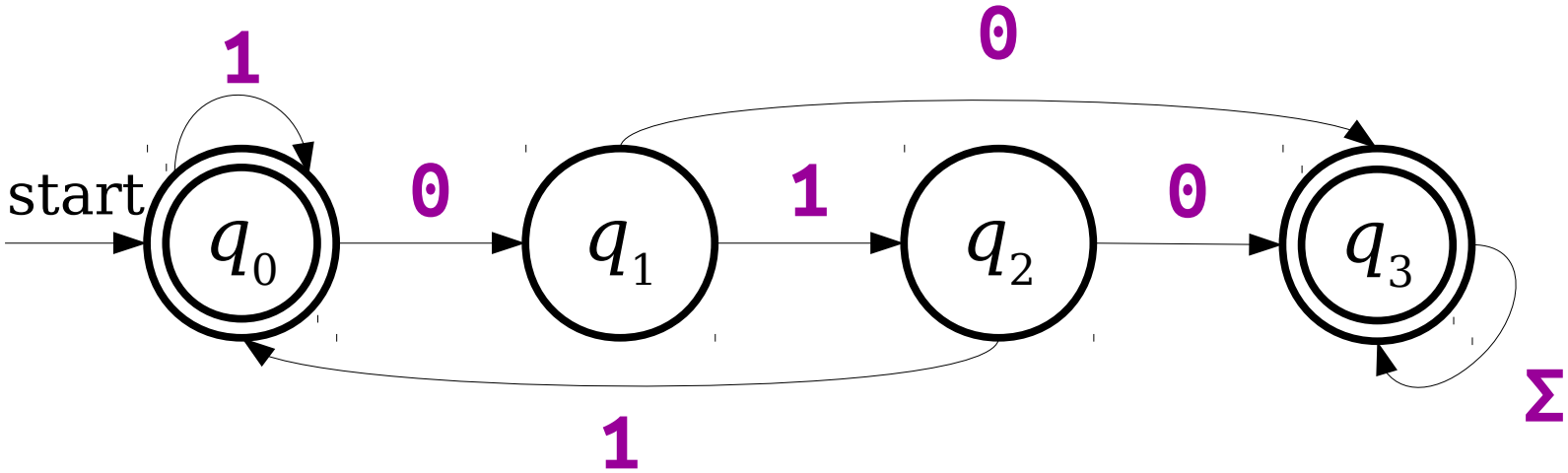
| $*q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_3$ | $q_2$ |
| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

# Tabular DFAs

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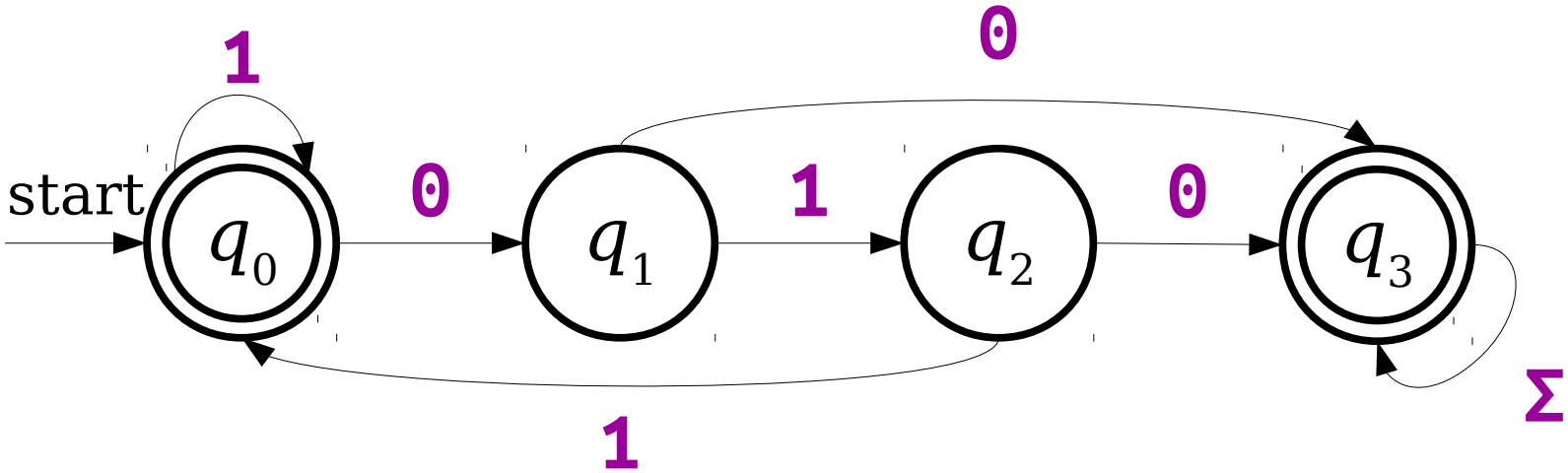
| $*q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_3$ | $q_2$ |
| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

These stars indicate accepting states.

# Tabular DFAs

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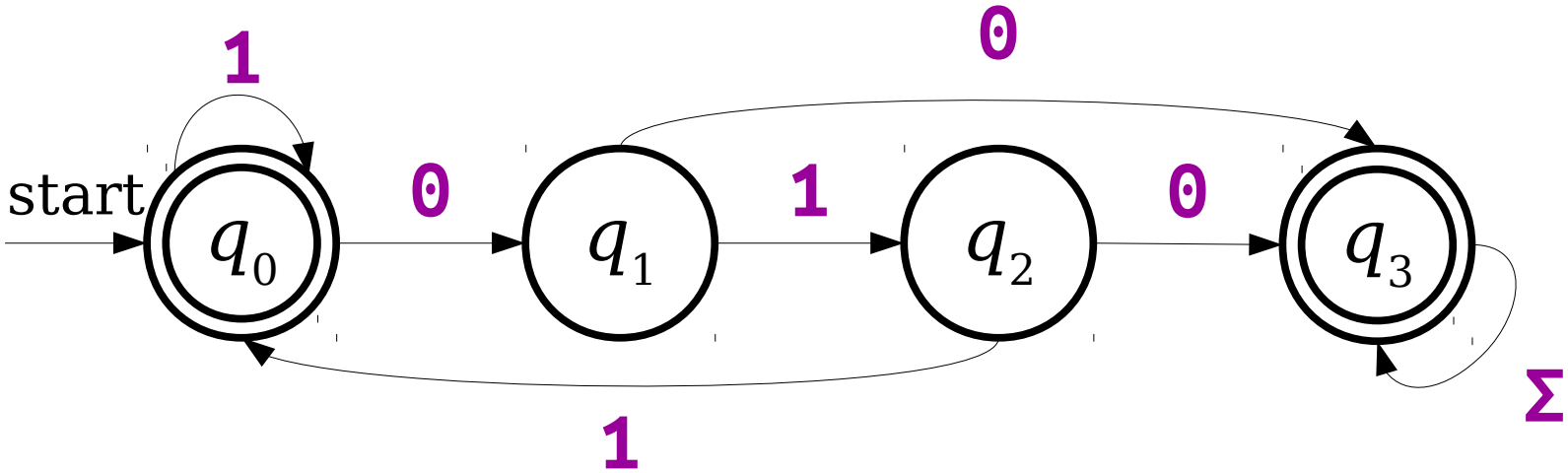
| $*q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_3$ | $q_2$ |
| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

Since this is the first row, it's the start state.

# Tabular DFAs

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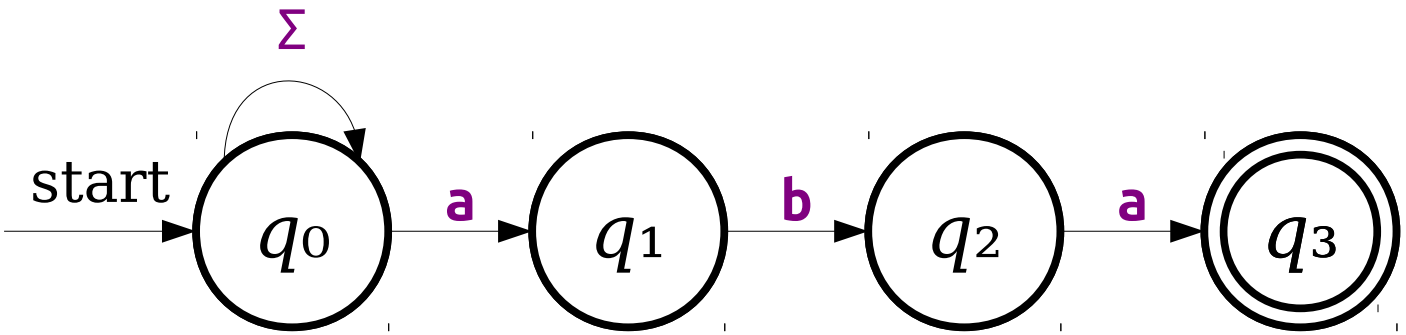
| $q_1$ | $q_3$ | $q_2$ |
| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

Question to ponder: Why isn't there a column here for Σ?

# Code? In a Theory Class?

```
int kTransitionTable[kNumStates][kNumSymbols] = {
     {0, 0, 1, 3, 7, 1, …},
      …
};
bool kAcceptTable[kNumStates] = {
    false,
    true,
    true,
    …
};
bool SimulateDFA(string input) {
    int state = 0;
    for (char ch: input) {
        state = kTransitionTable[state][ch];
    }
    return kAcceptTable[state];
}
```

Can we do something similar for NFAs?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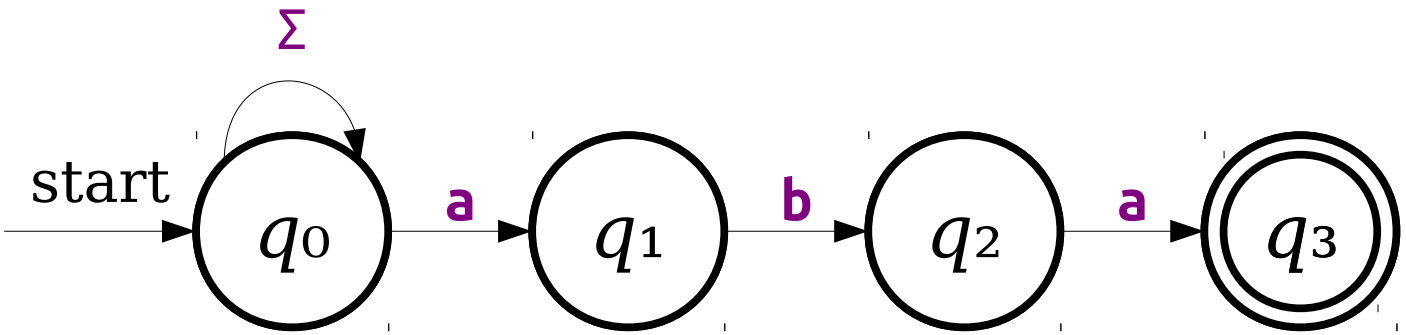
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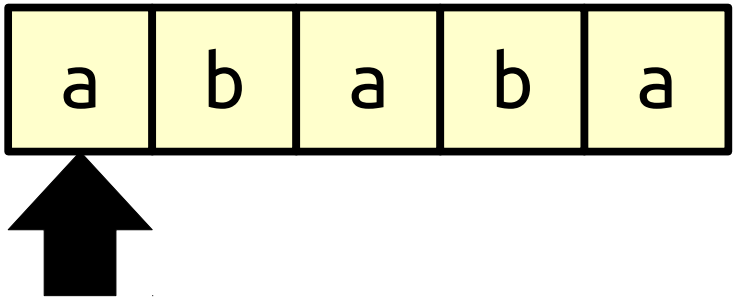
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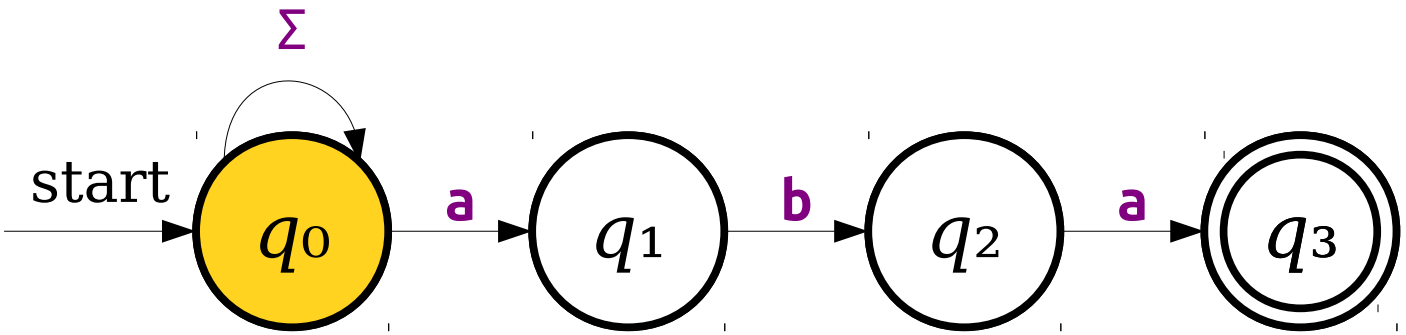
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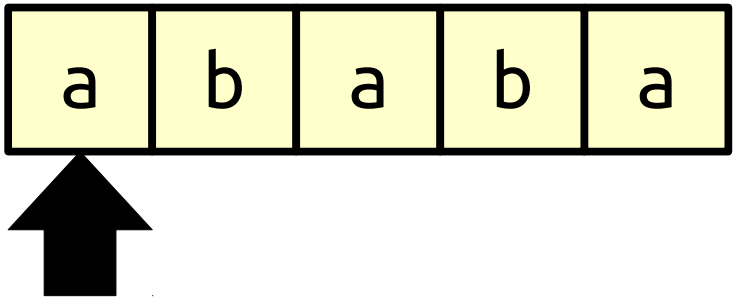
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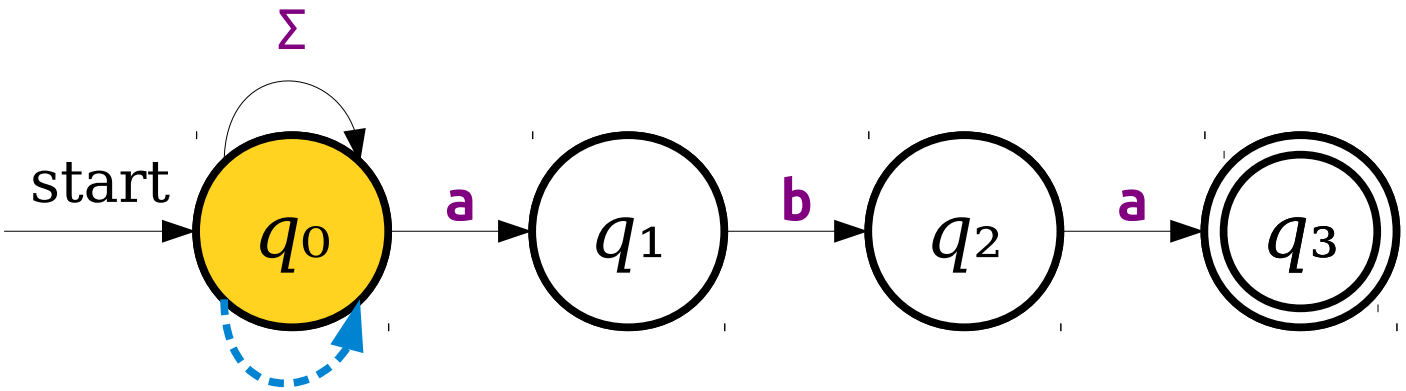
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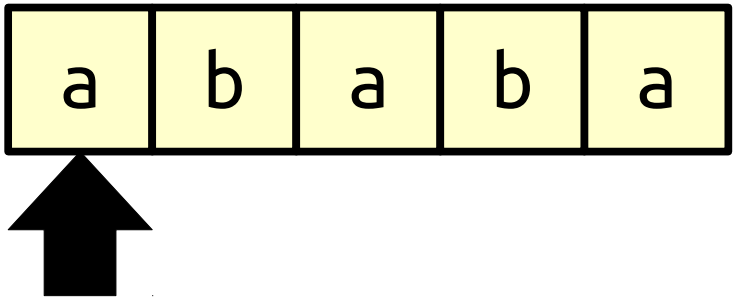
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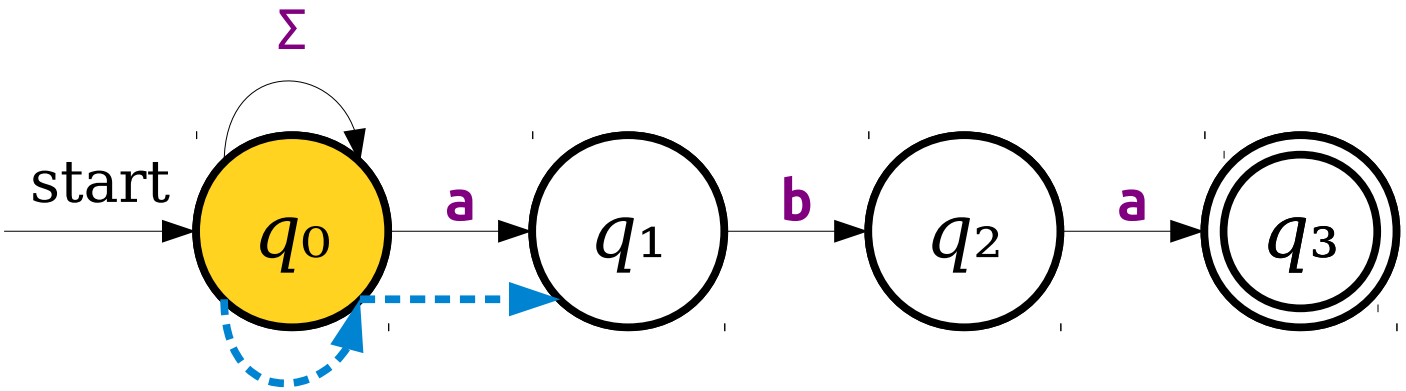
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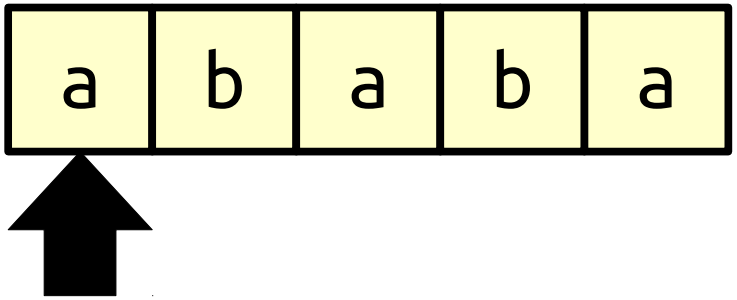
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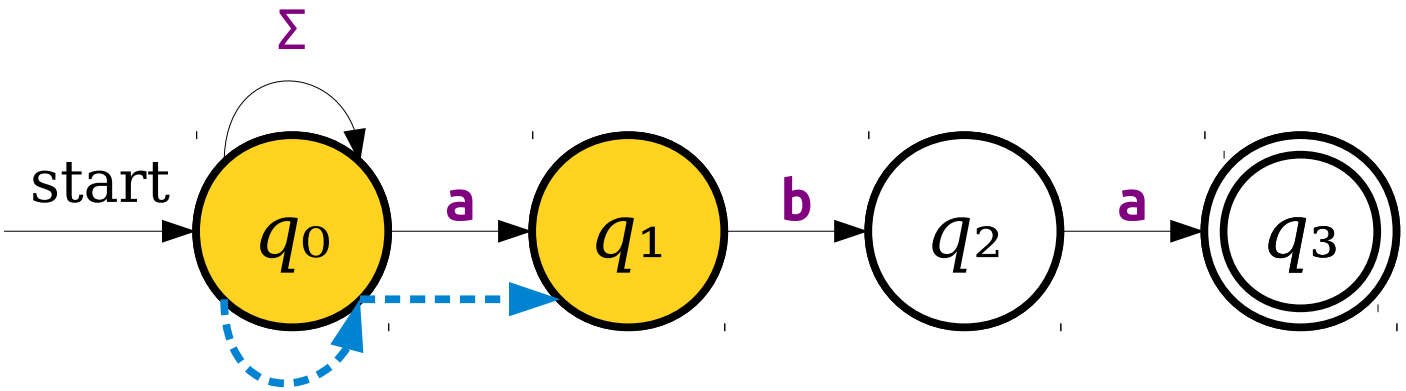
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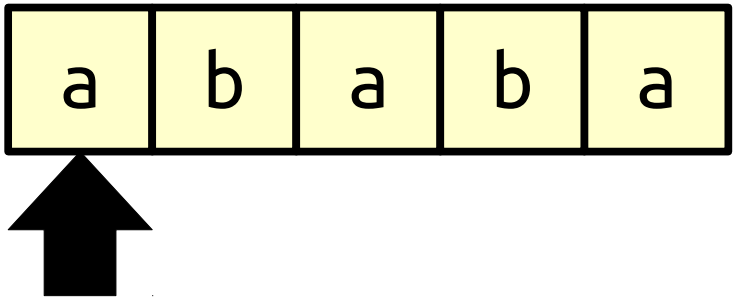
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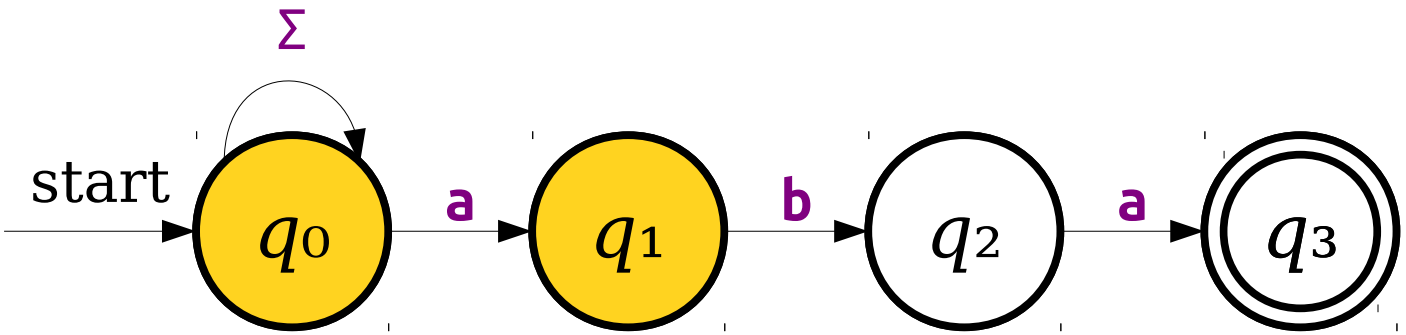
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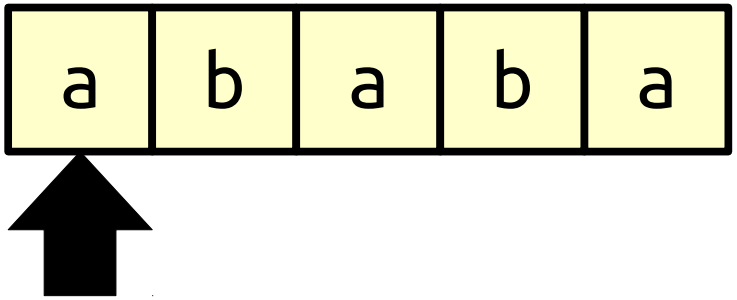
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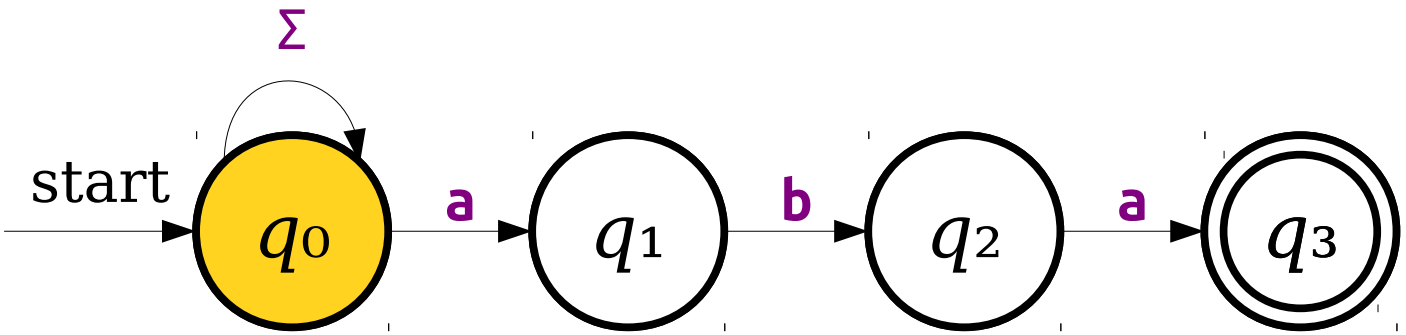


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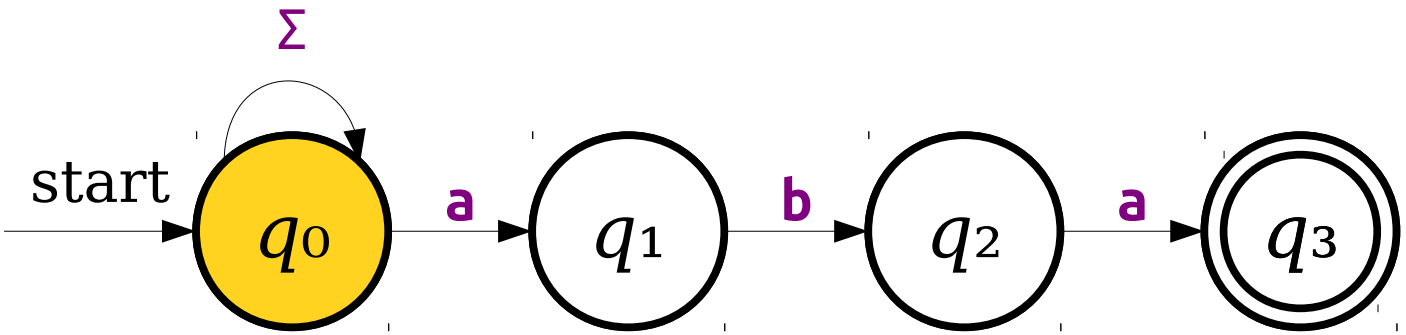
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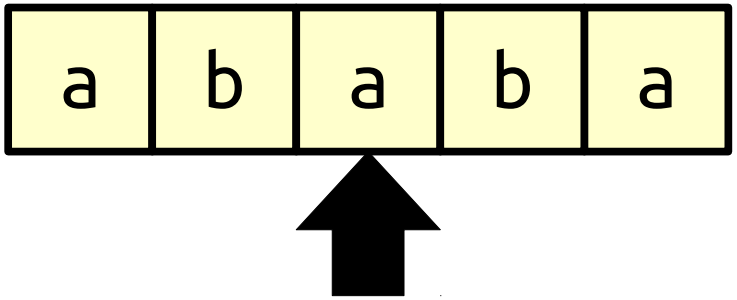
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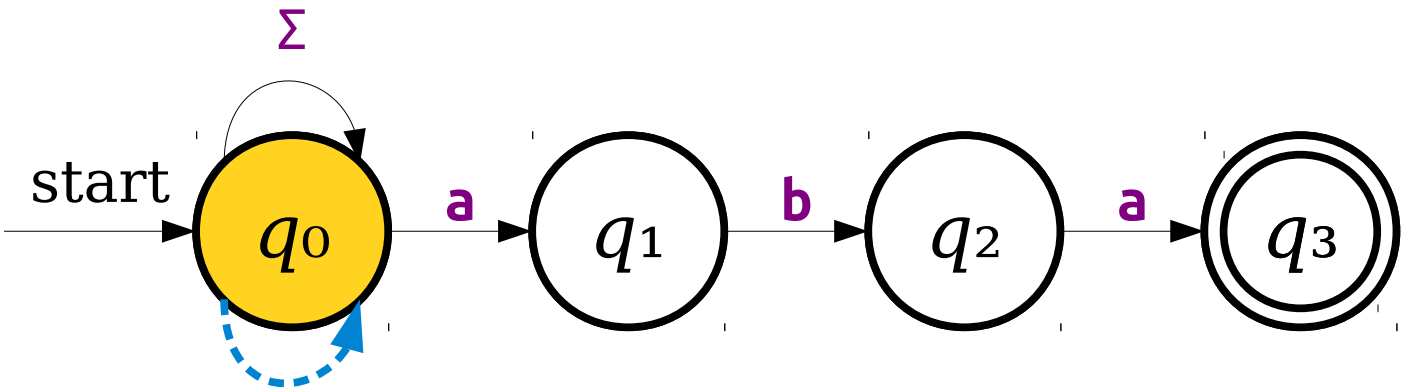
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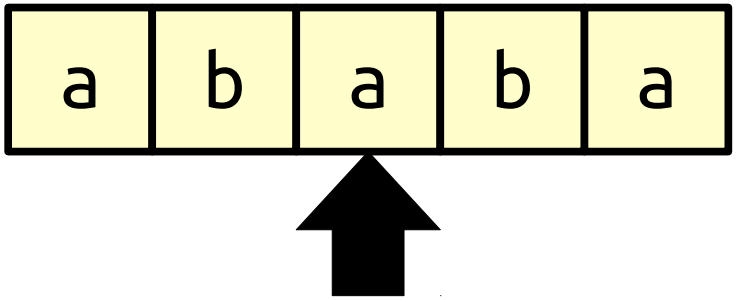
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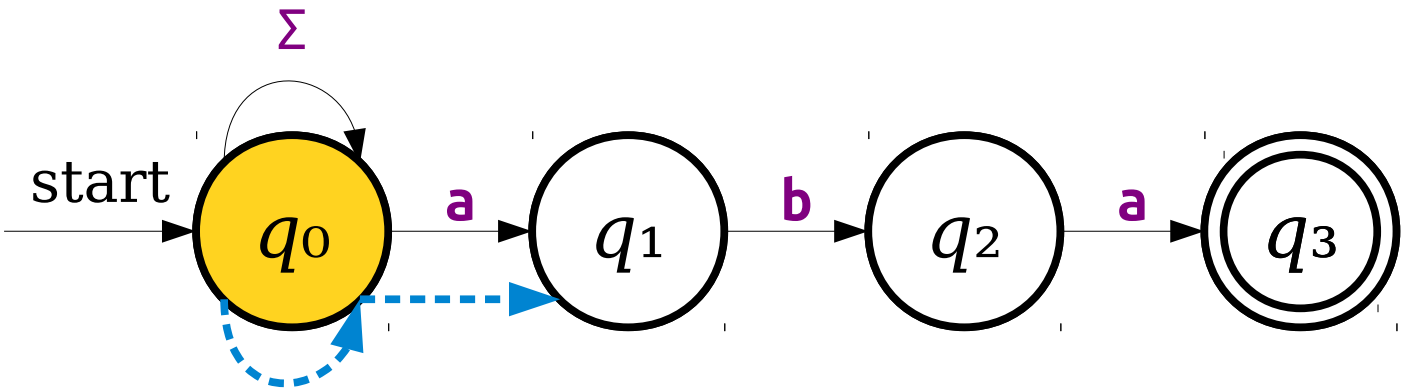
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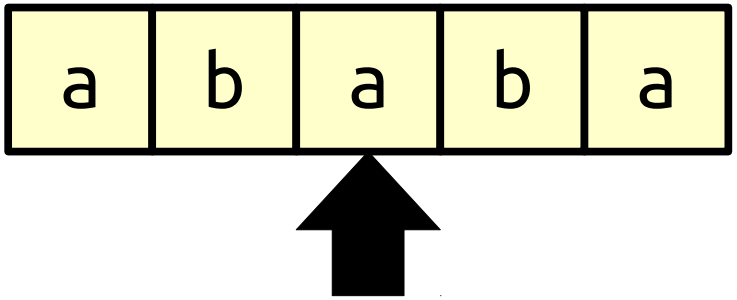
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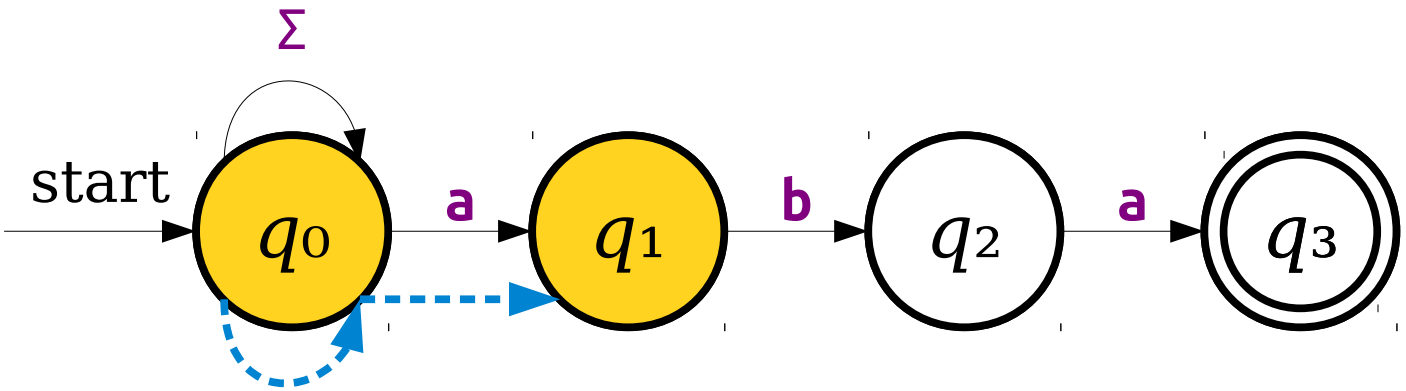
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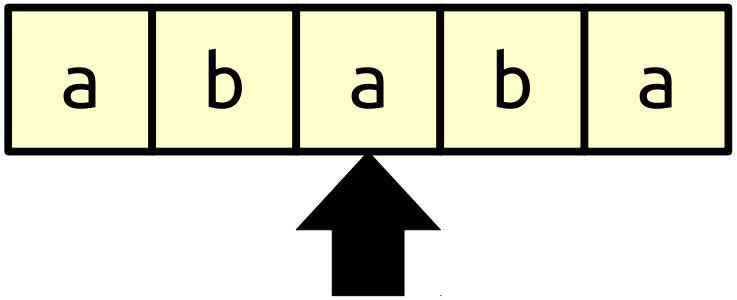
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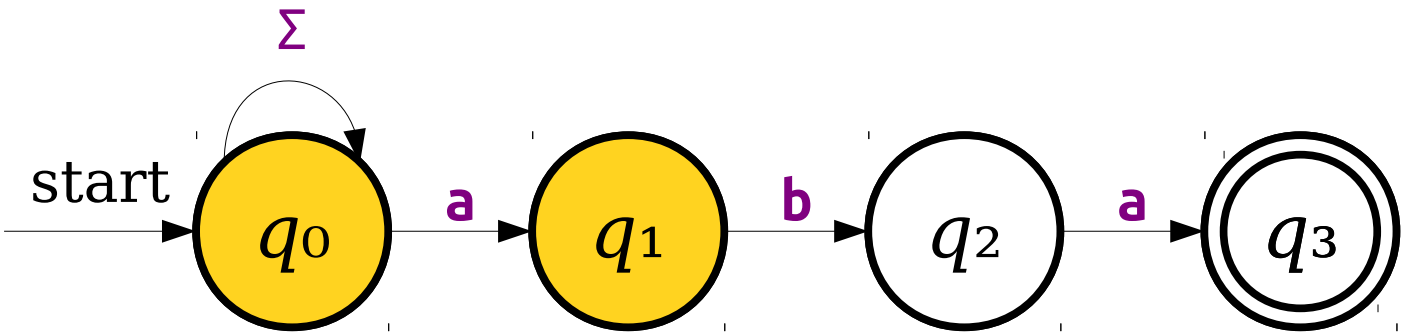
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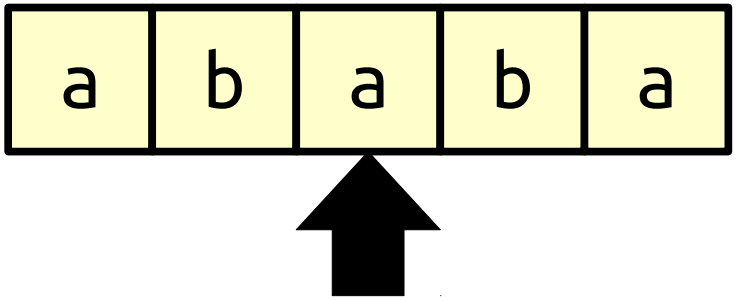
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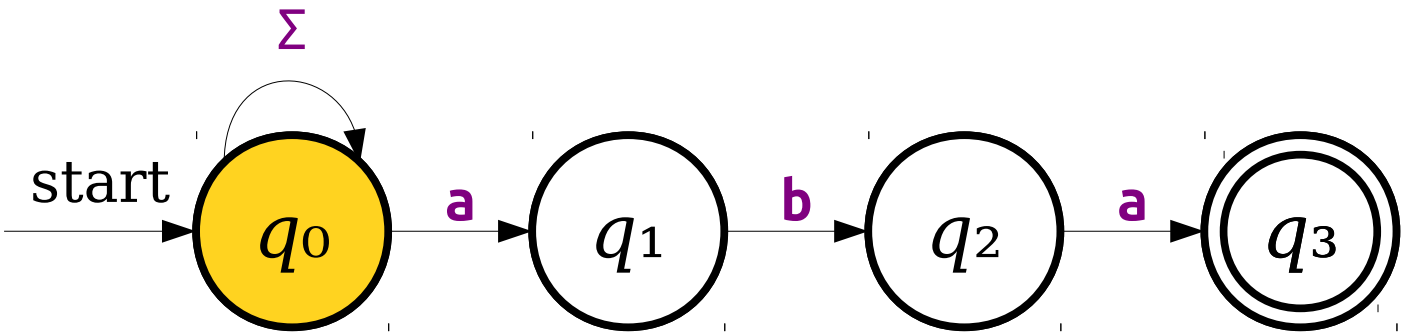


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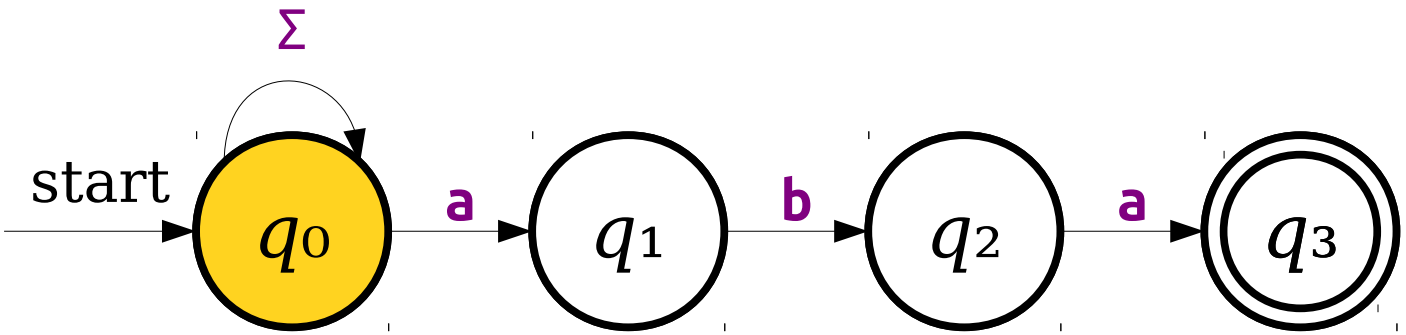
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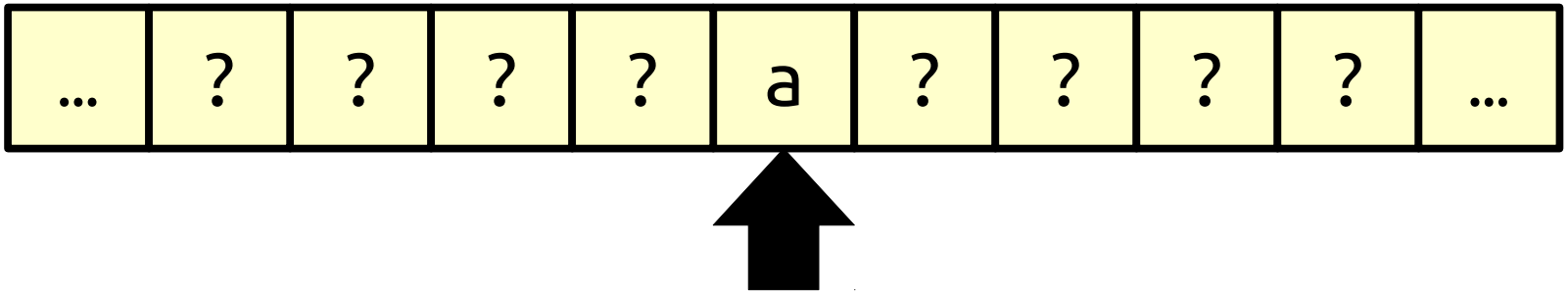
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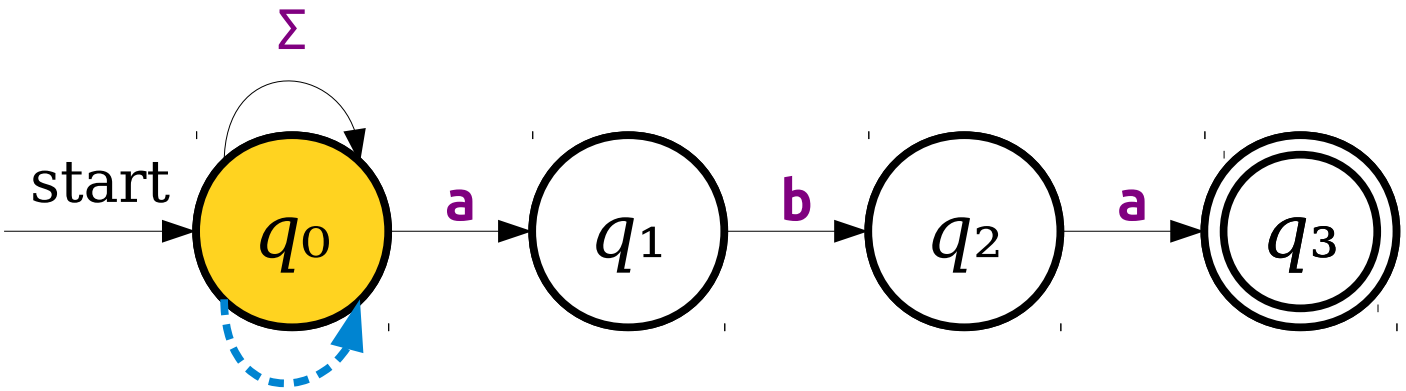
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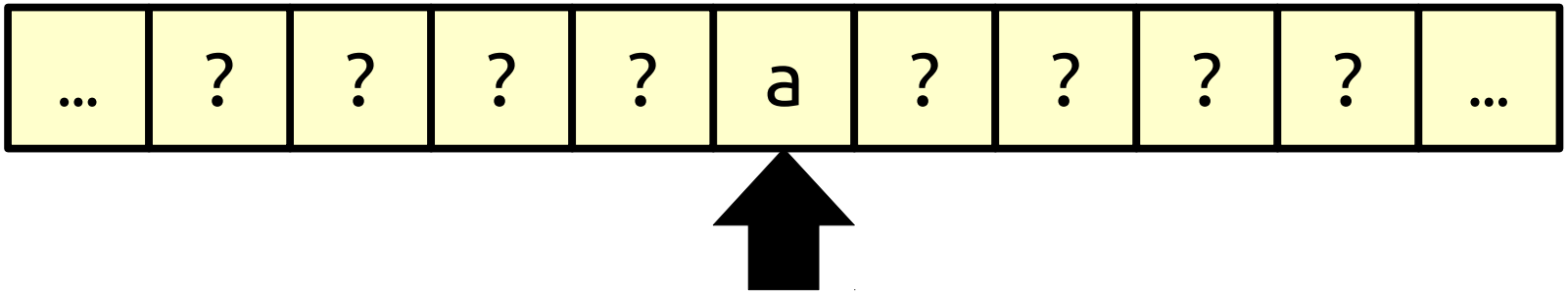
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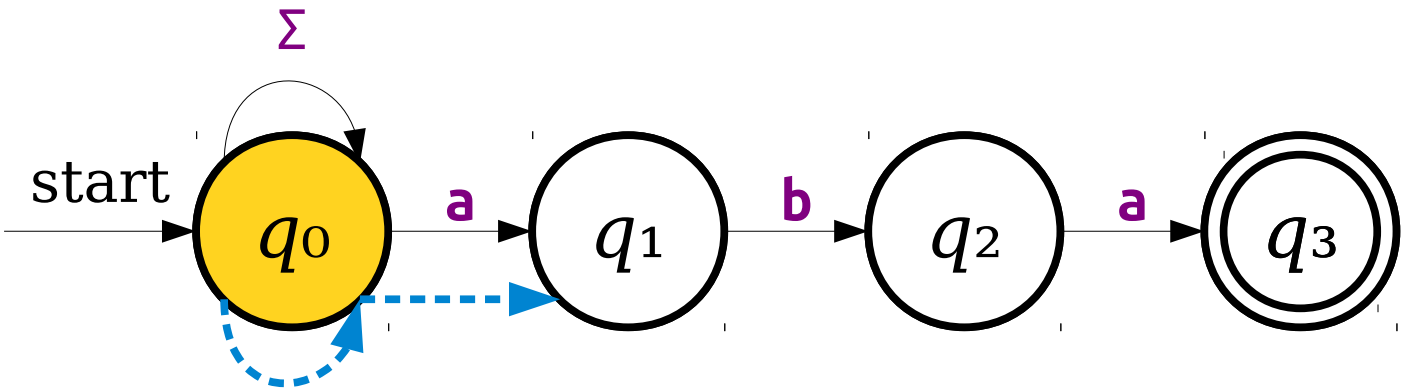
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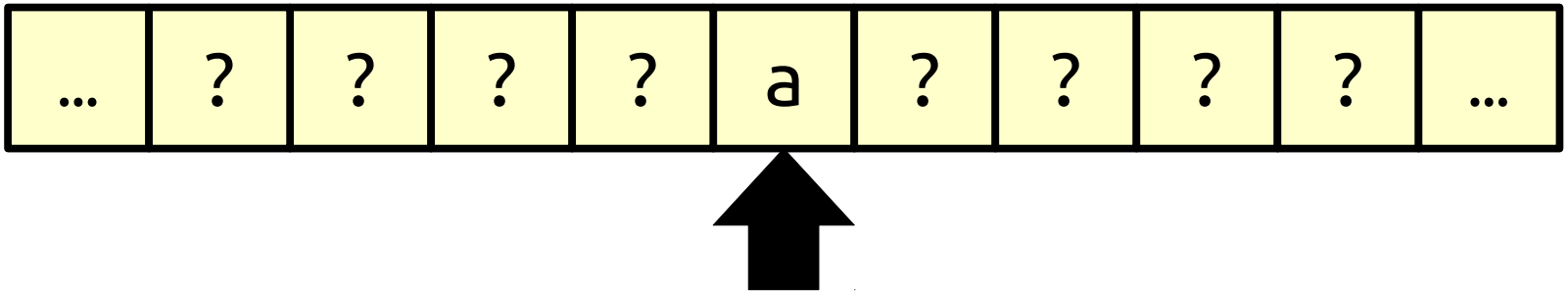
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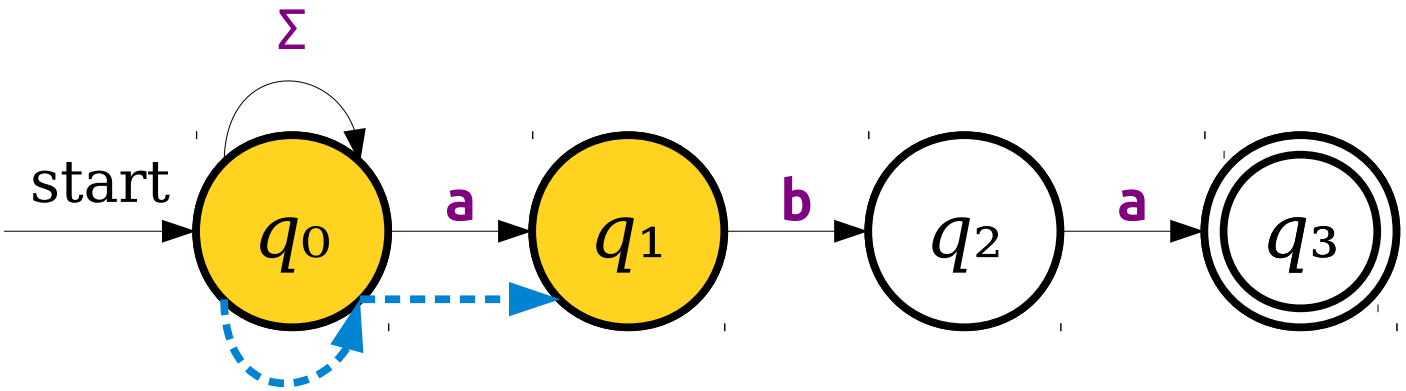
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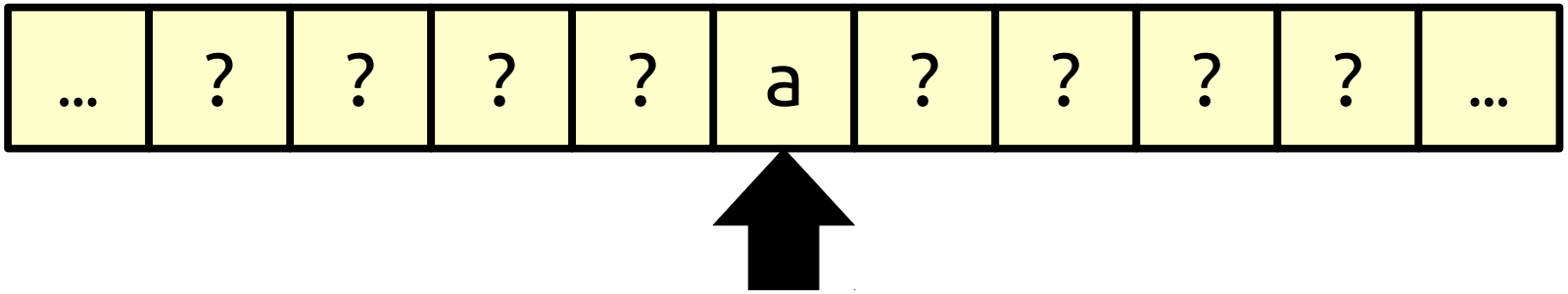
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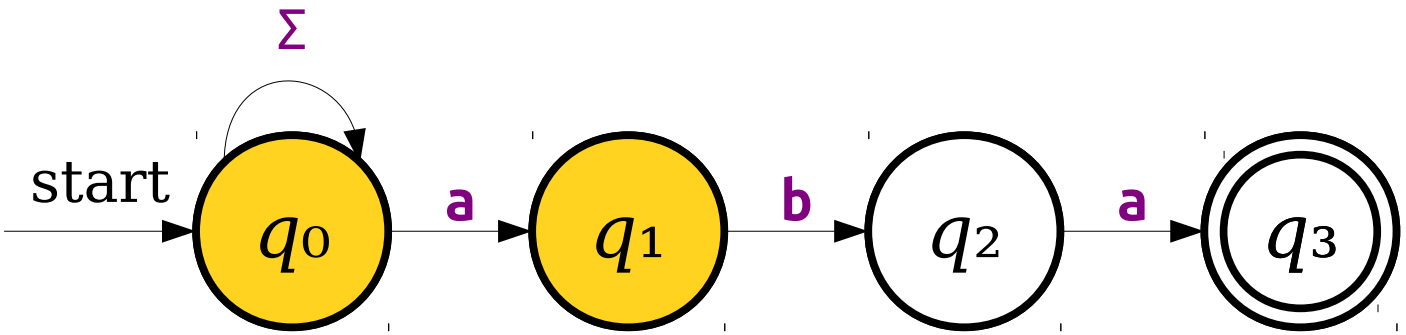
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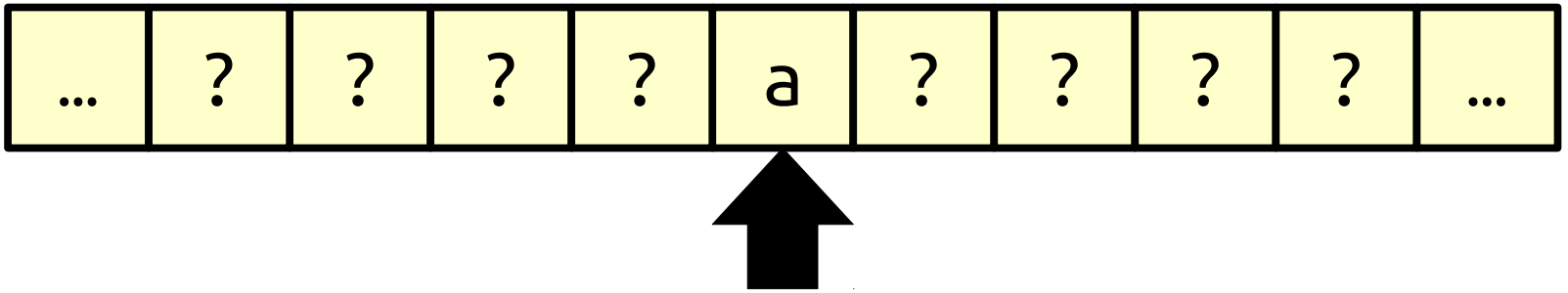
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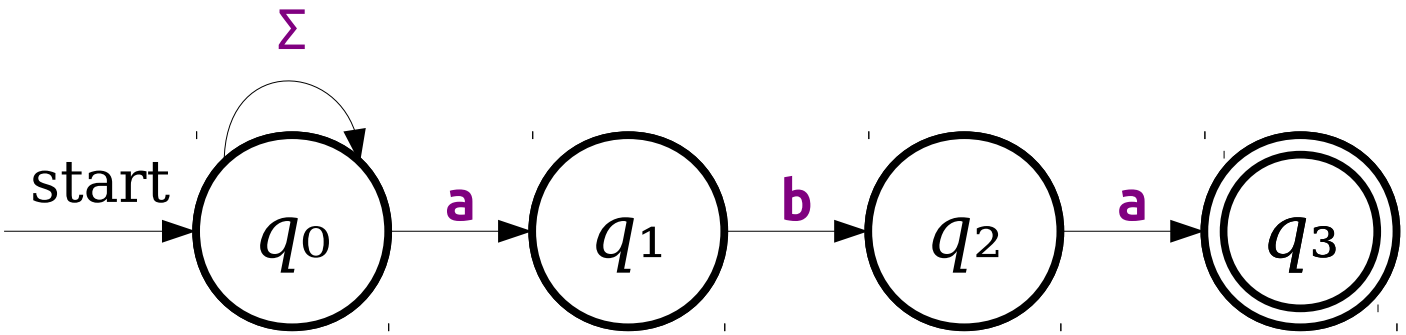


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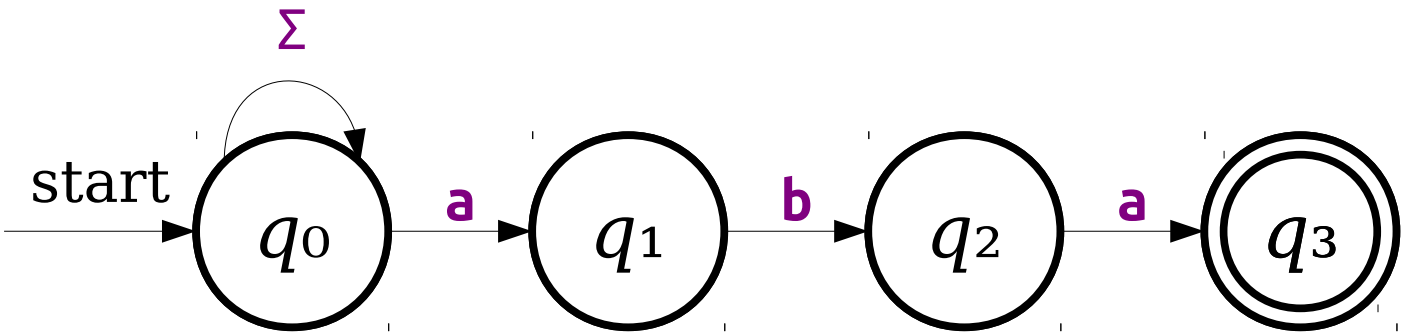


| | $a$ | $b$ |
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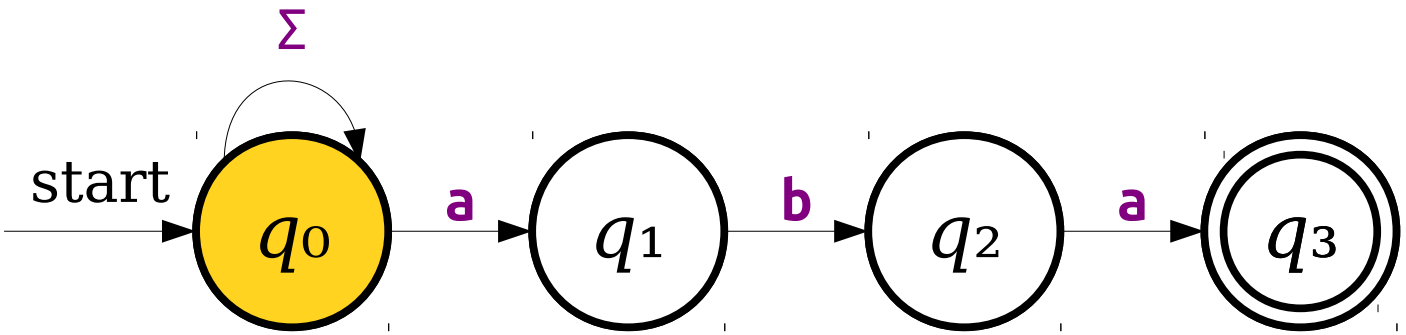


| | $a$ | $b$ |
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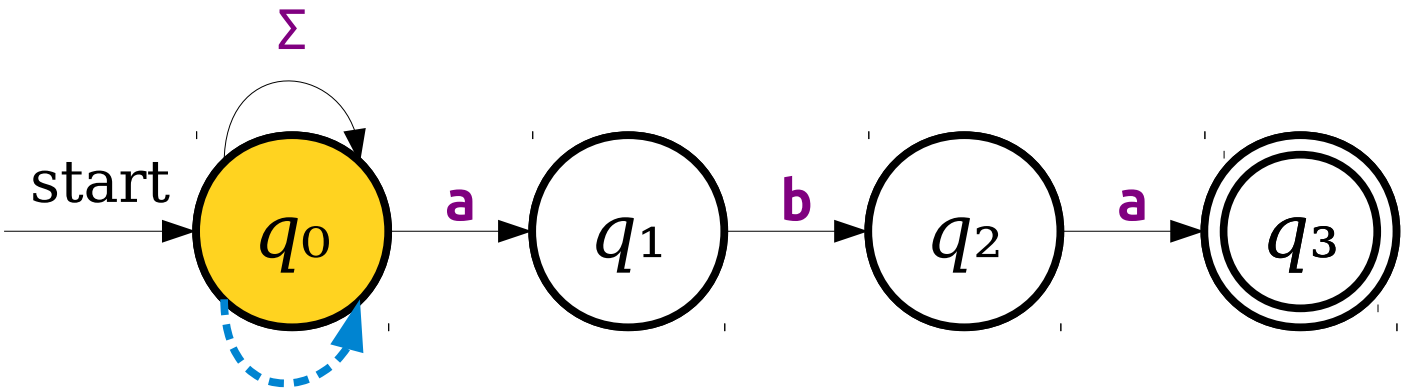


| | $a$ | $b$ |
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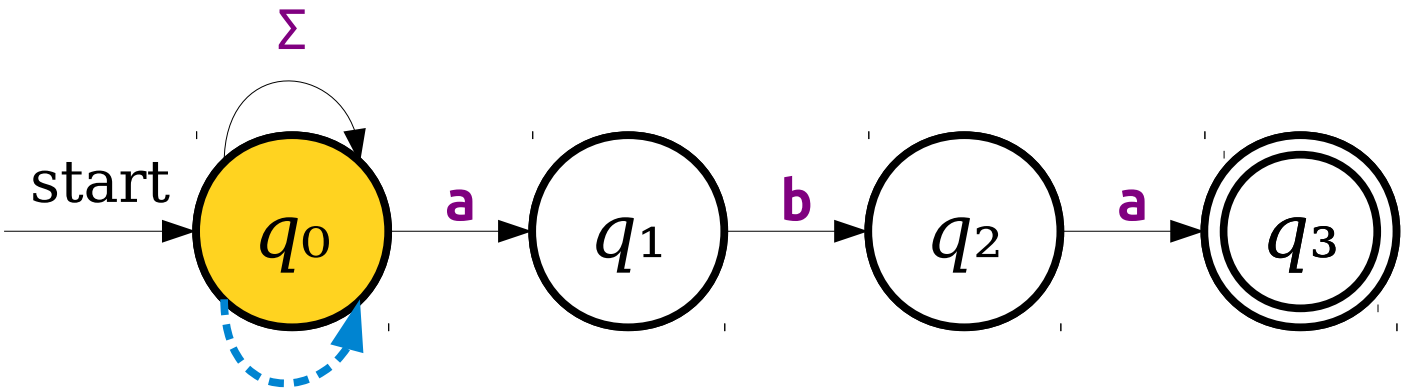


| | $a$ | $b$ |
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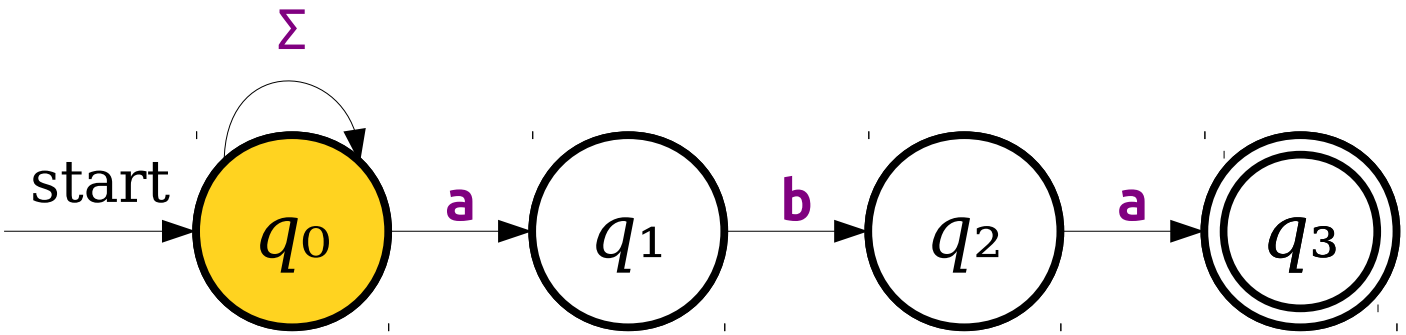


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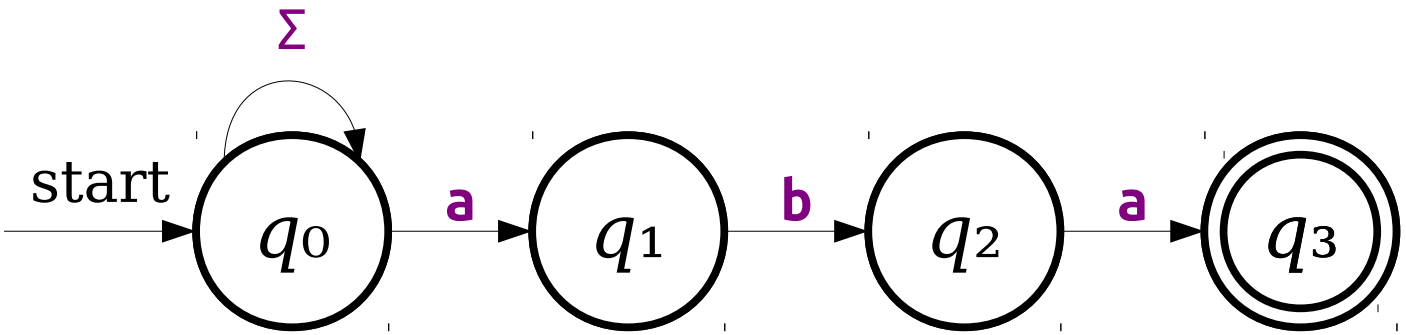


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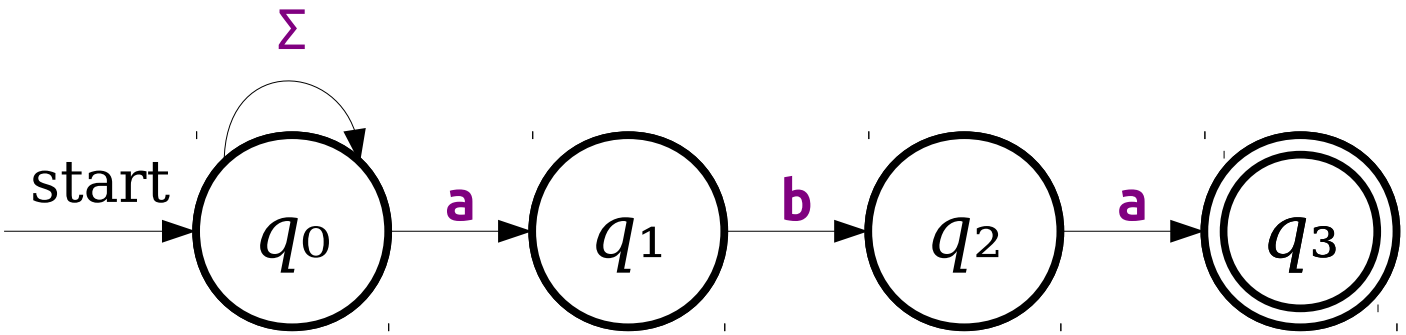


| | $a$ | $b$ |
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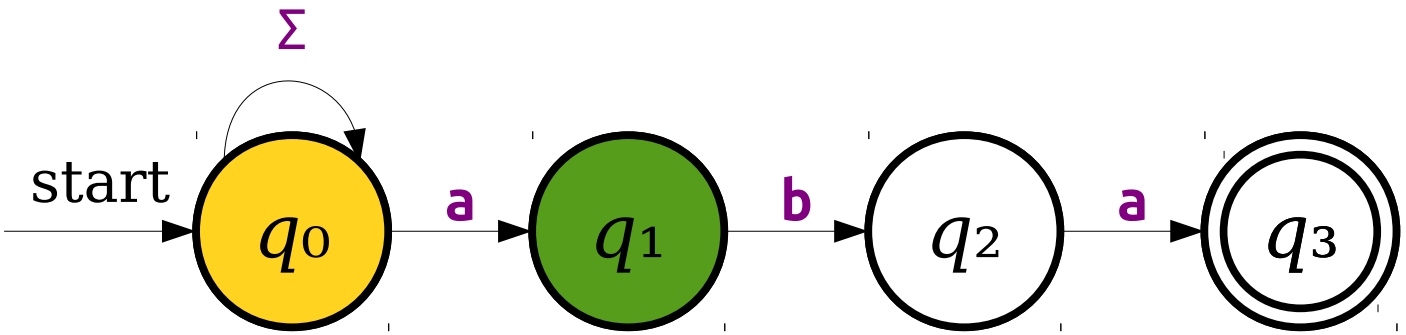


| | $a$ | $b$ |
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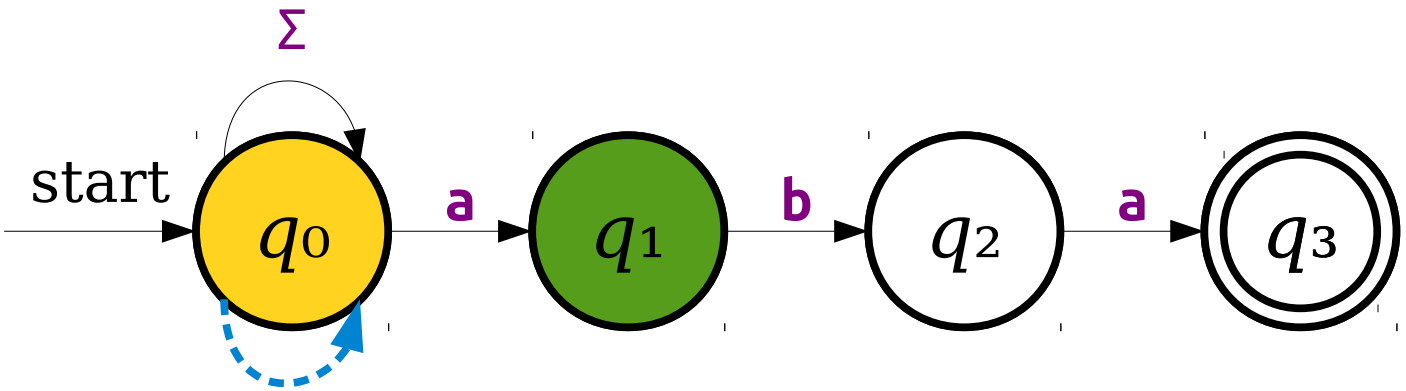


| | $a$ | $b$ |
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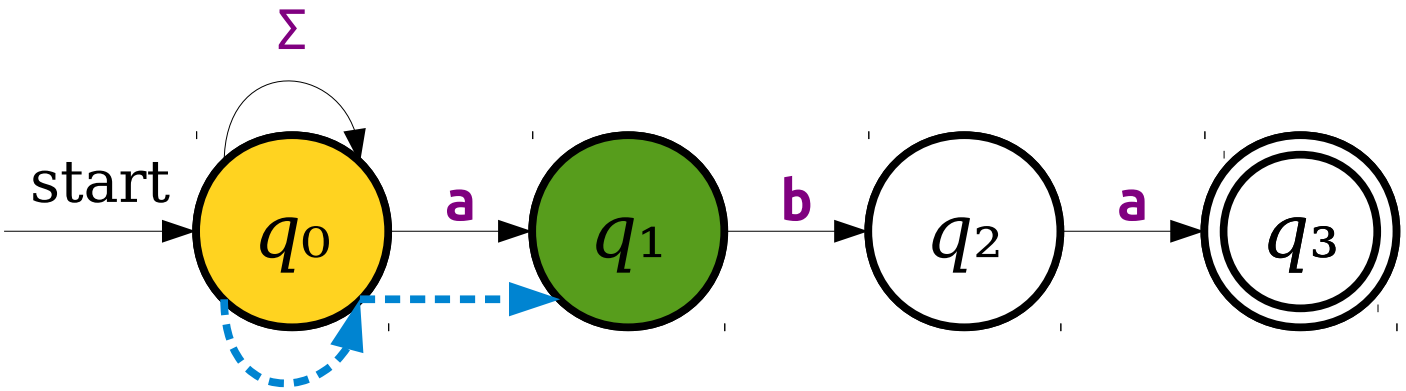


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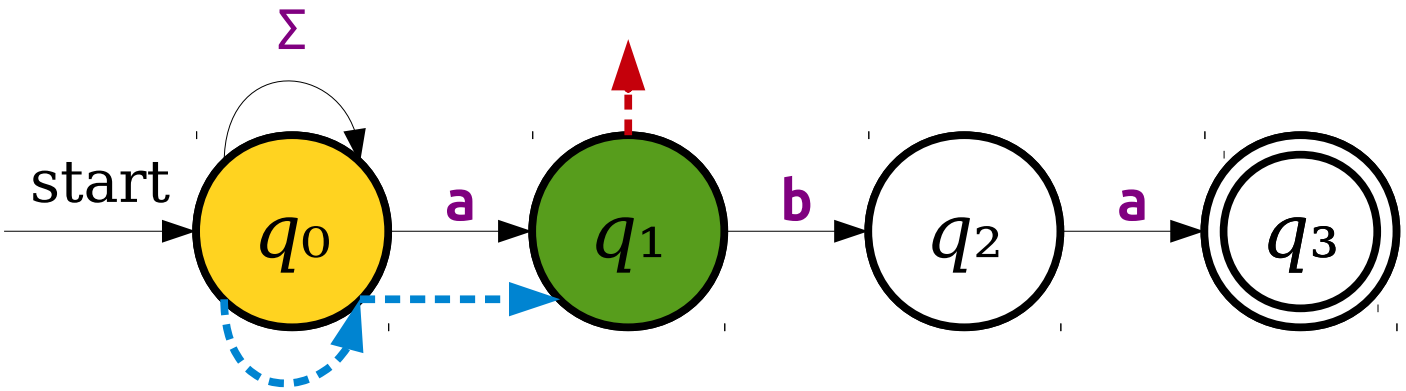


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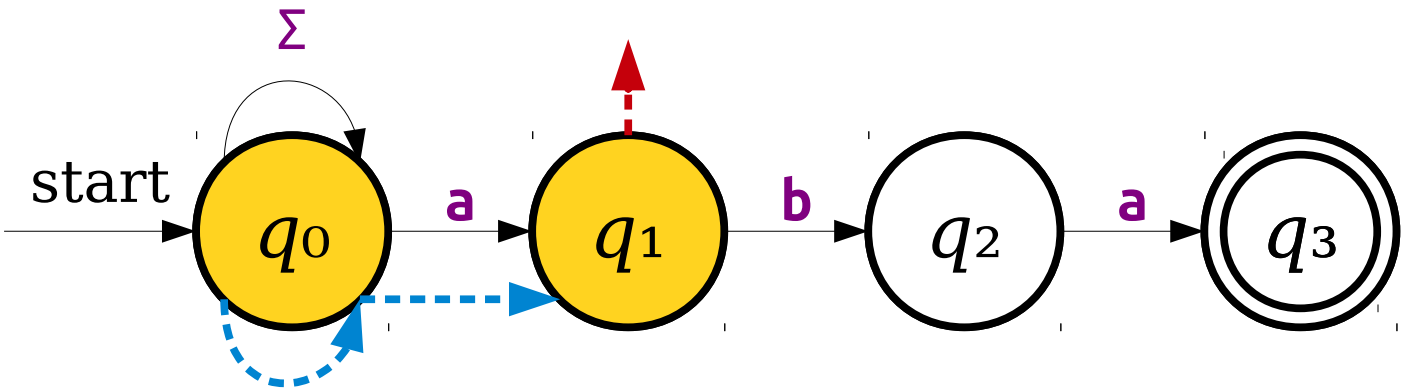


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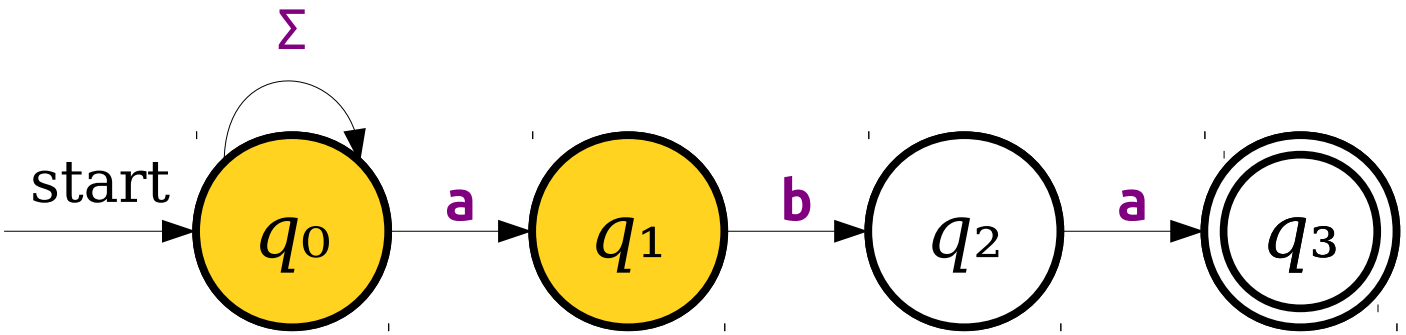


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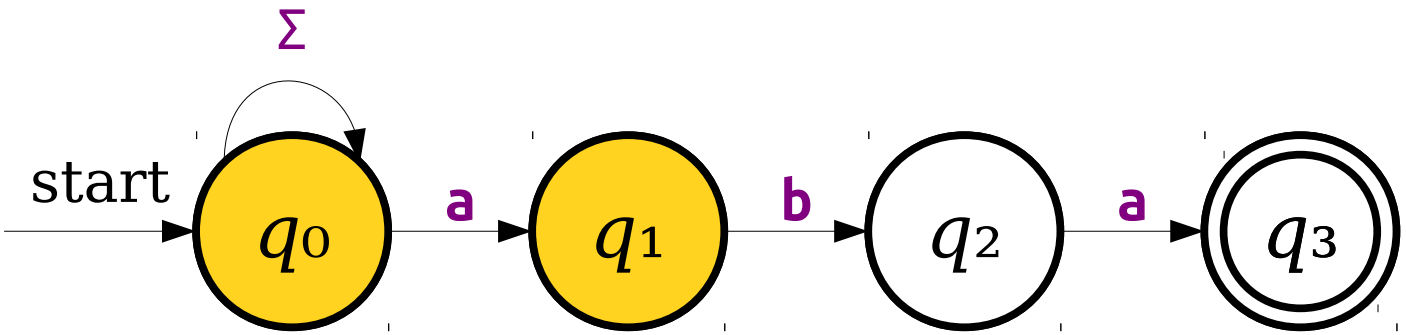


| | $a$ | $b$ |
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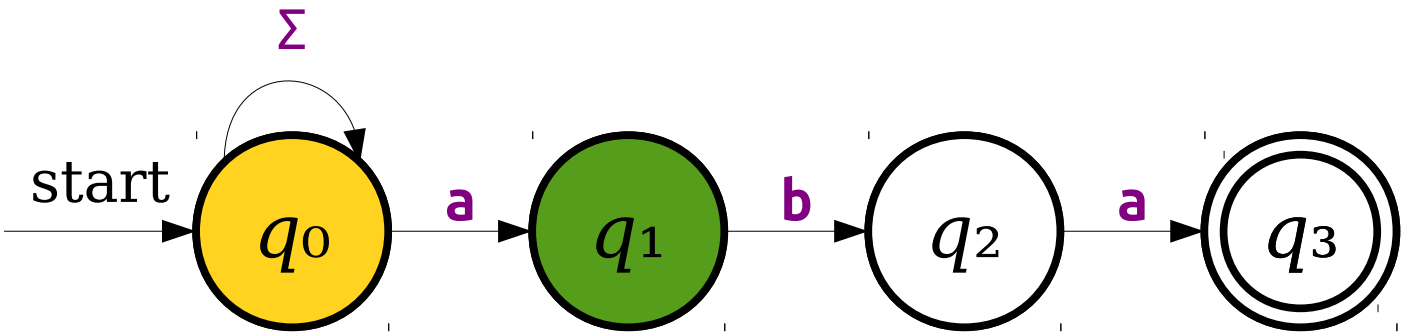


| | $a$ | $b$ |
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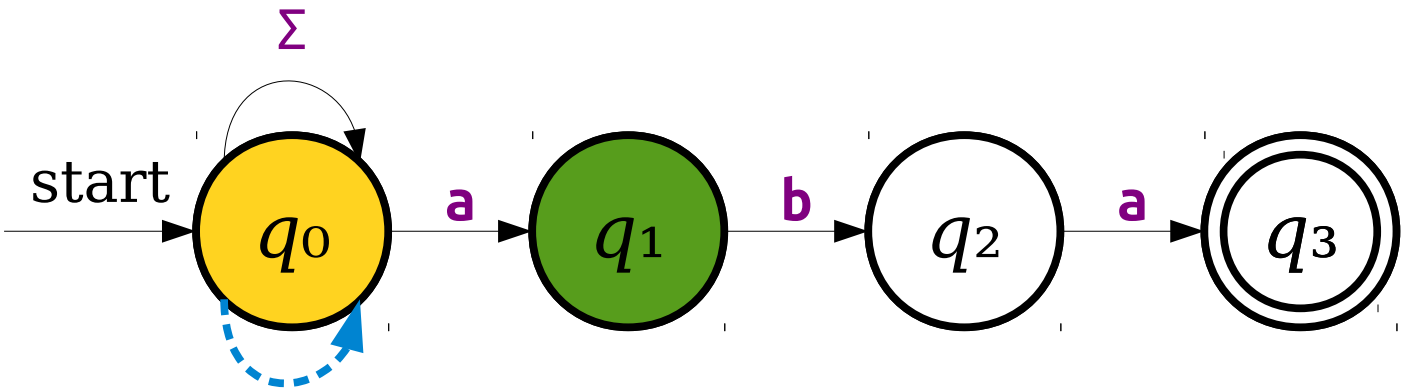


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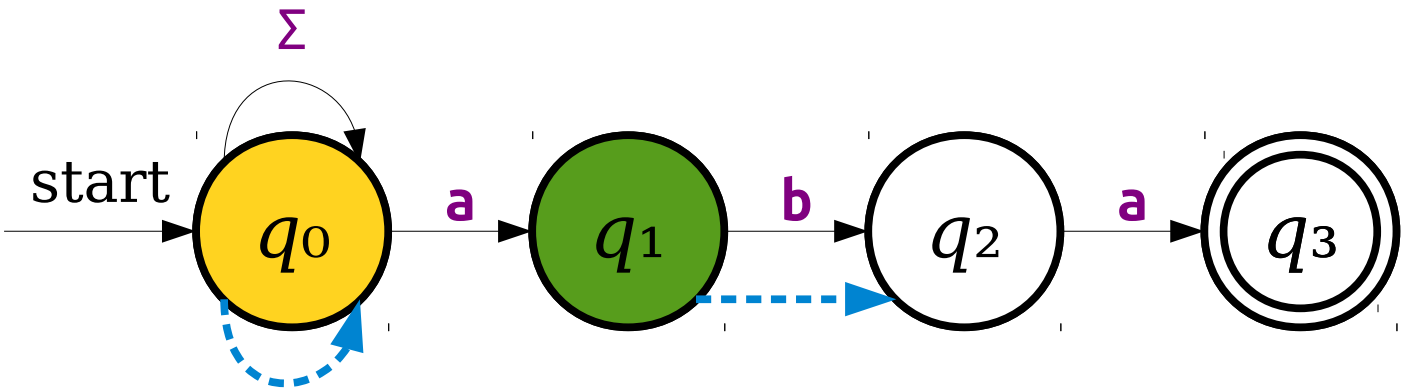


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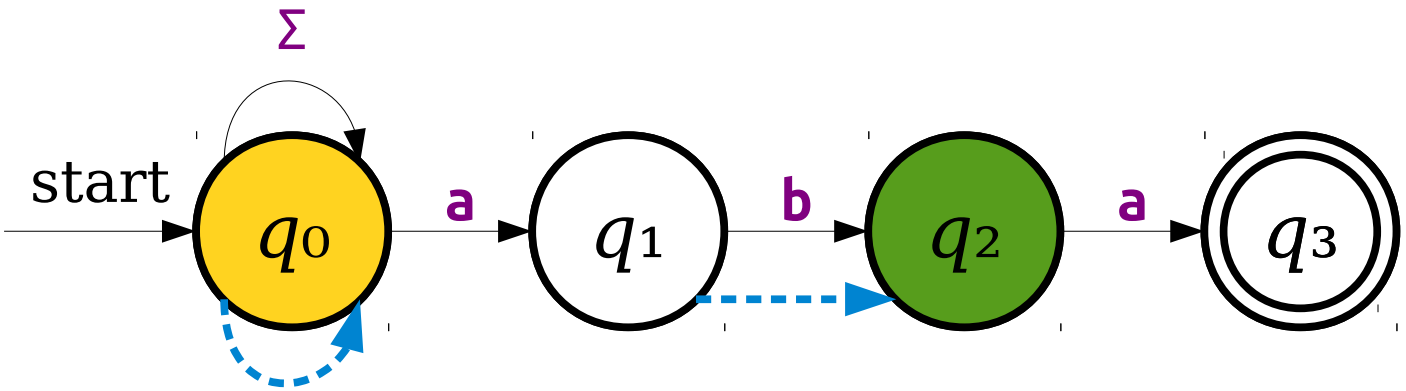


| | $a$ | $b$ |
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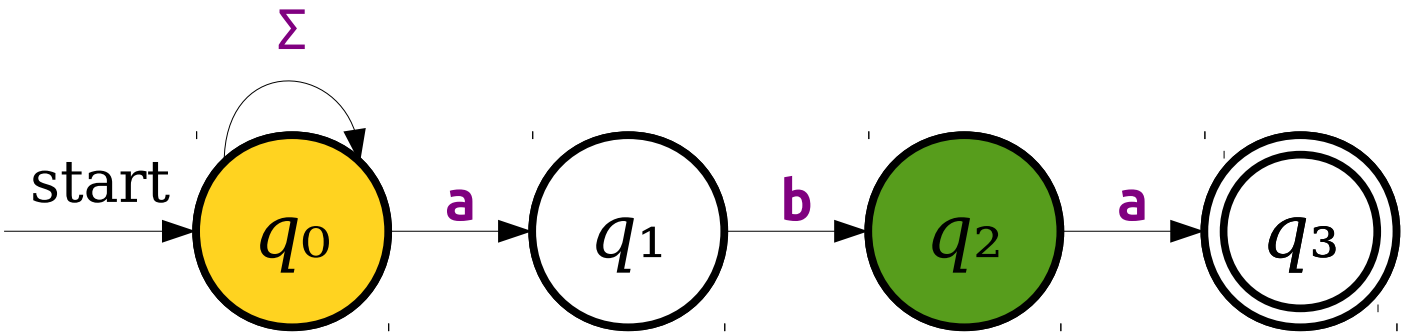


| | $a$ | $b$ |
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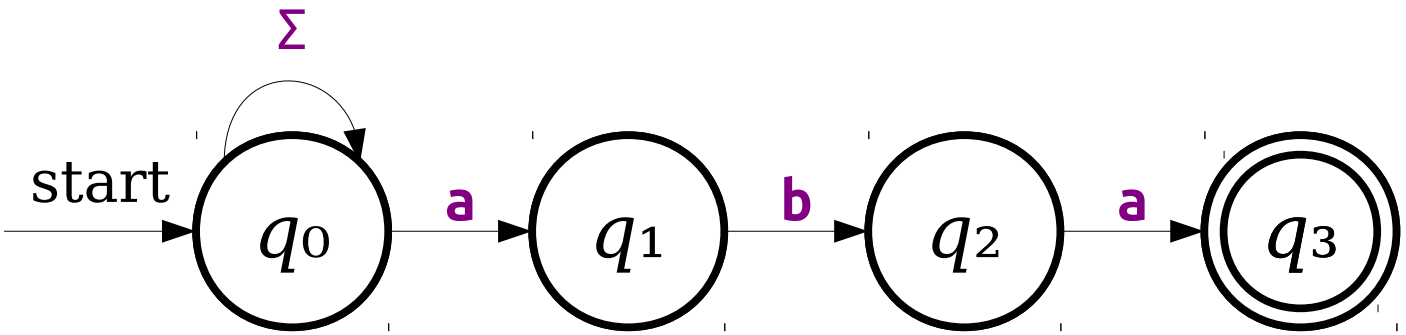


| | $a$ | $b$ |
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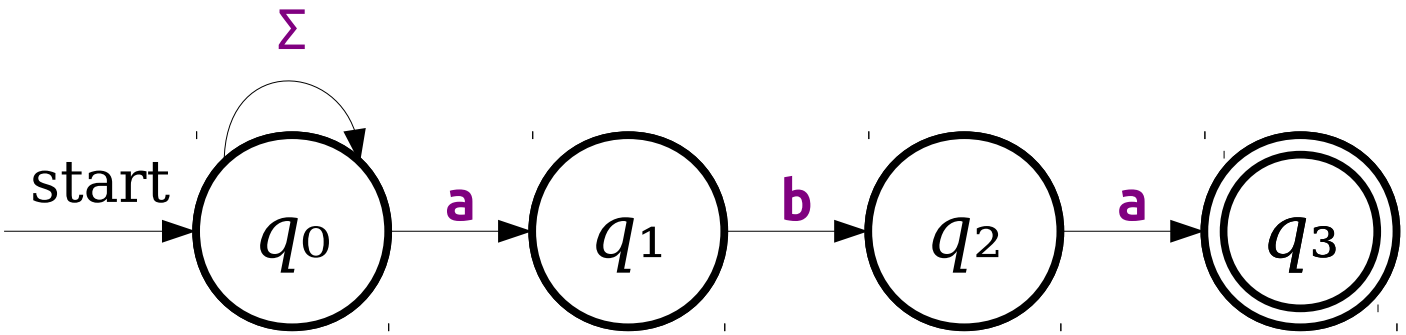


| | $a$ | $b$ |
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| $\{q_0\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
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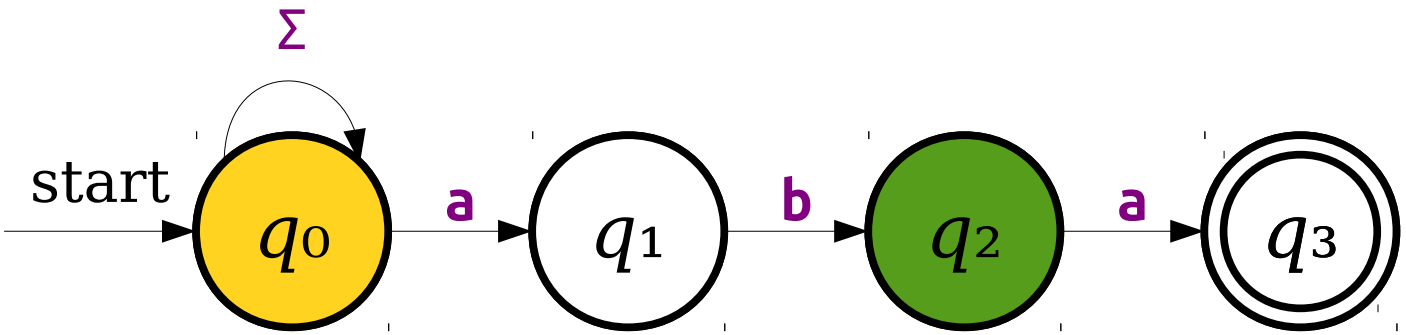


| | $a$ | $b$ |
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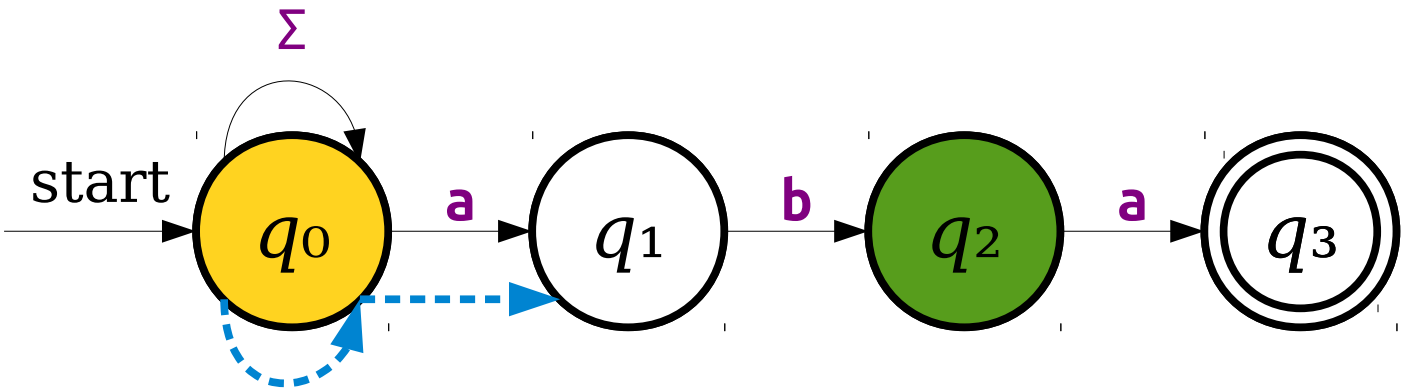


| | $a$ | $b$ |
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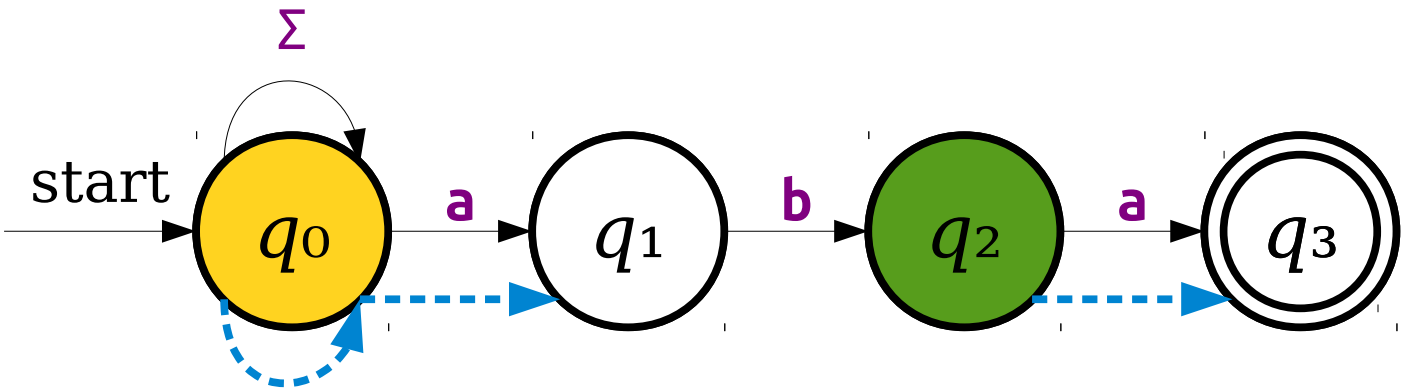


| | $a$ | $b$ |
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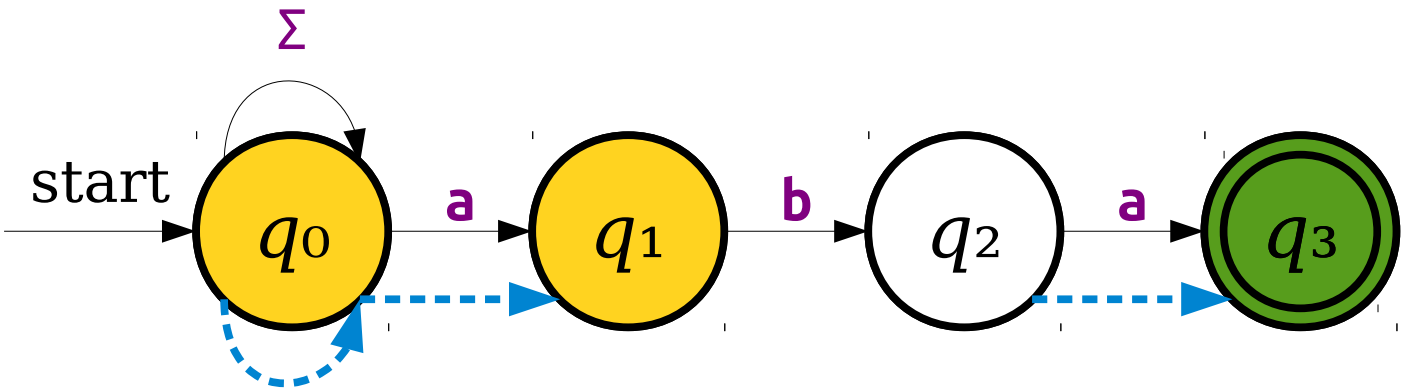


| | $a$ | $b$ |
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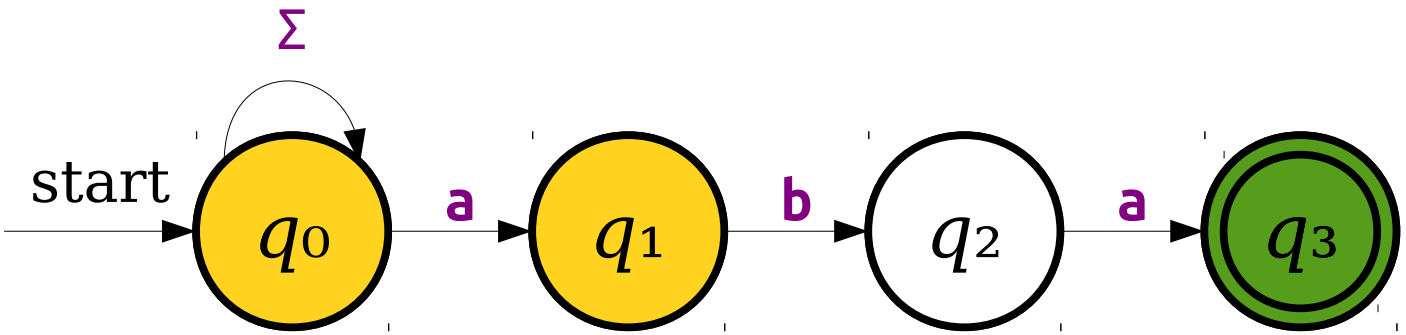


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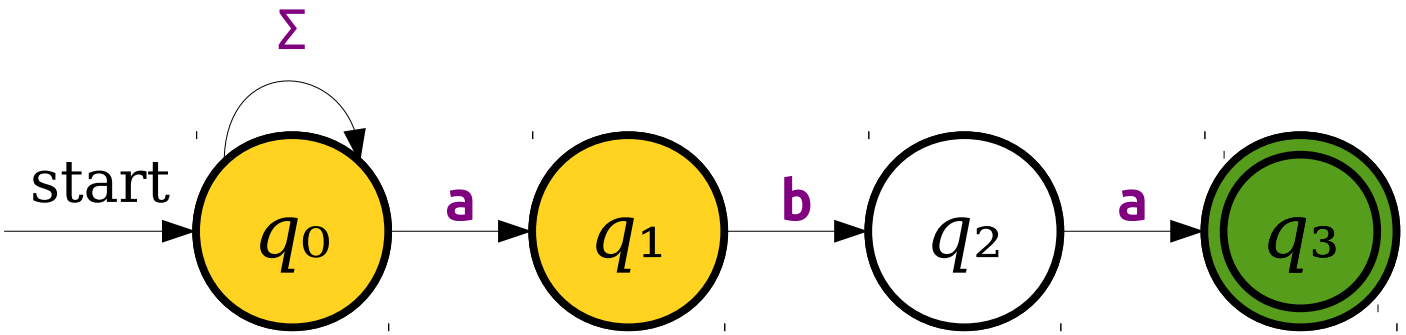


| | $a$ | $b$ |
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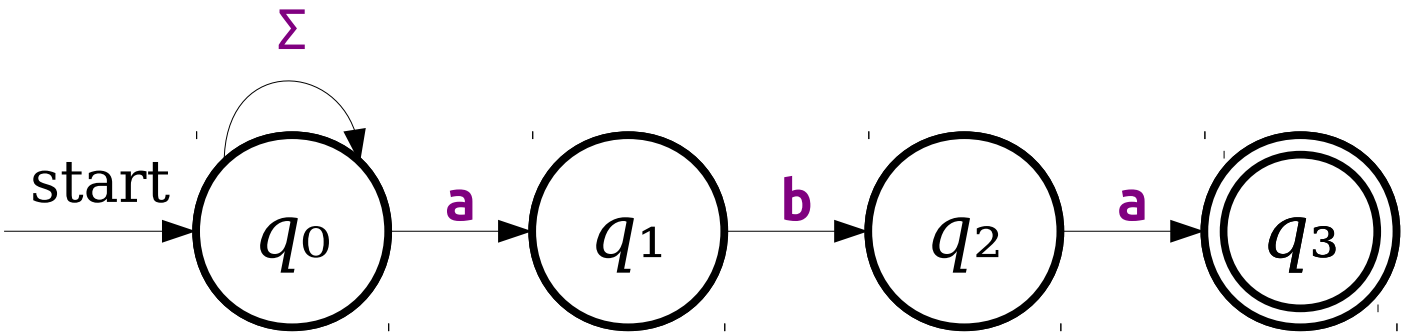


| | $a$ | $b$ |
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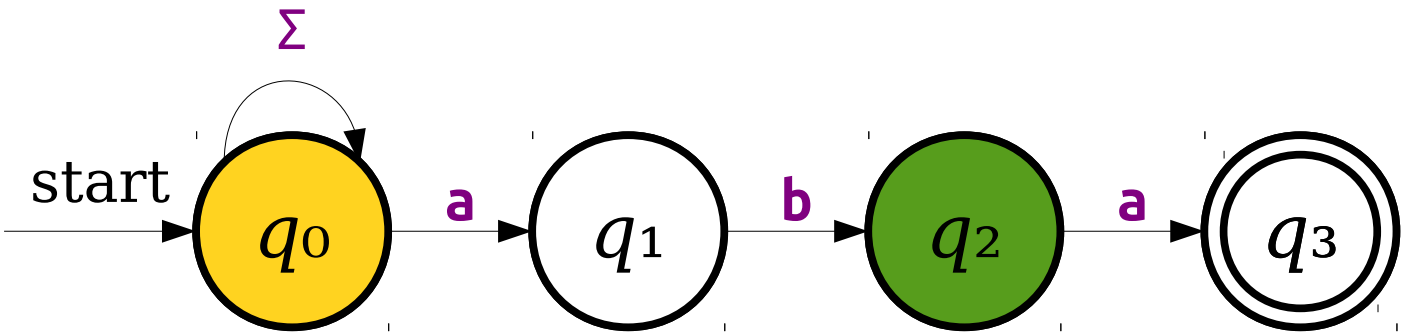


| | $a$ | $b$ |
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| $\{q_0, q_2\}$ | $\{q_0, q_1, q_3\}$ | |
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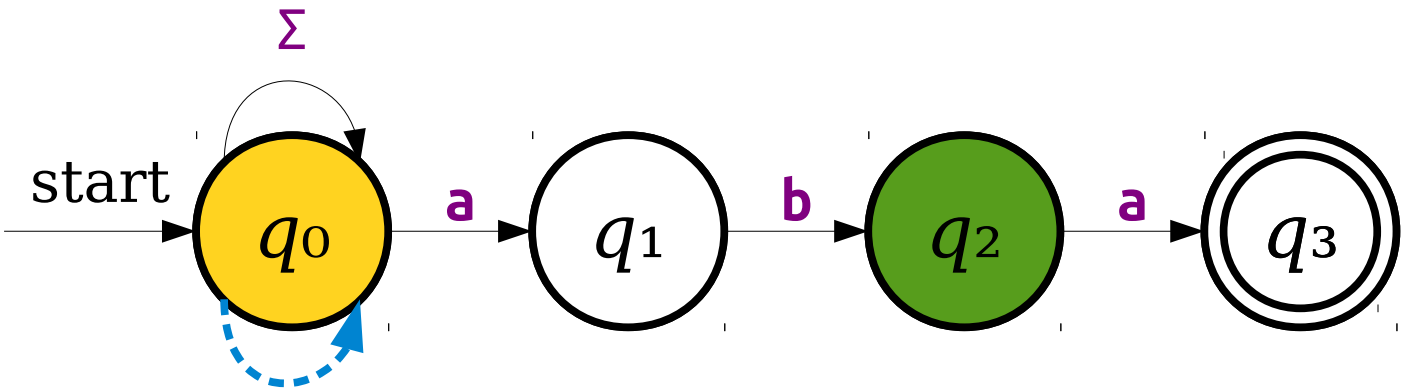


| | $a$ | $b$ |
|---|---|---|
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| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| $\{q_0, q_2\}$ | $\{q_0, q_1, q_3\}$ | |
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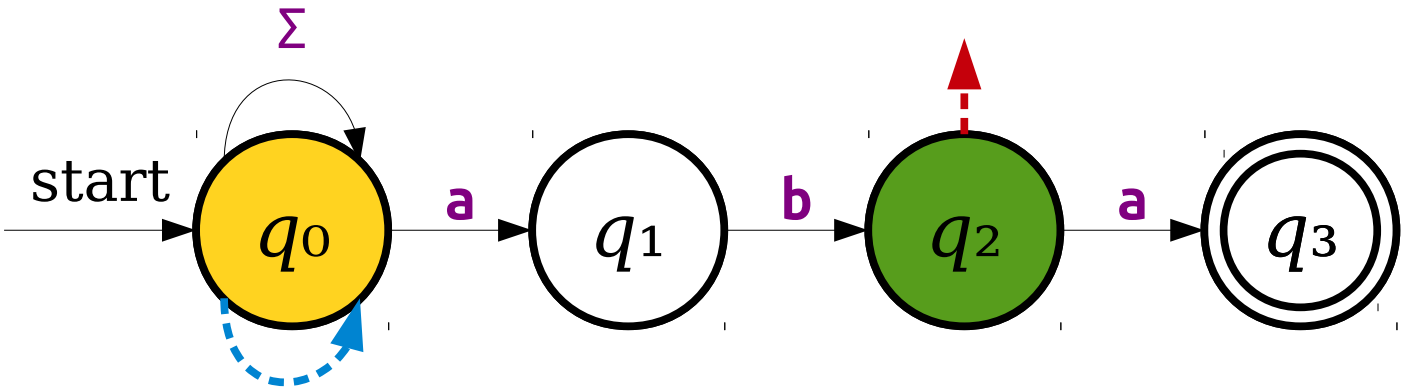


| | $a$ | $b$ |
|---|---|---|
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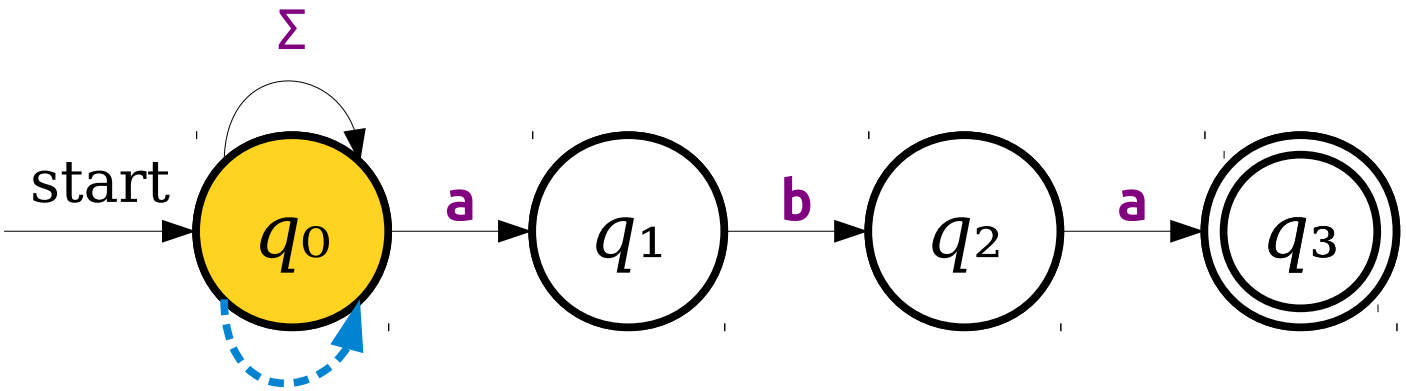


| | $a$ | $b$ |
|---|---|---|
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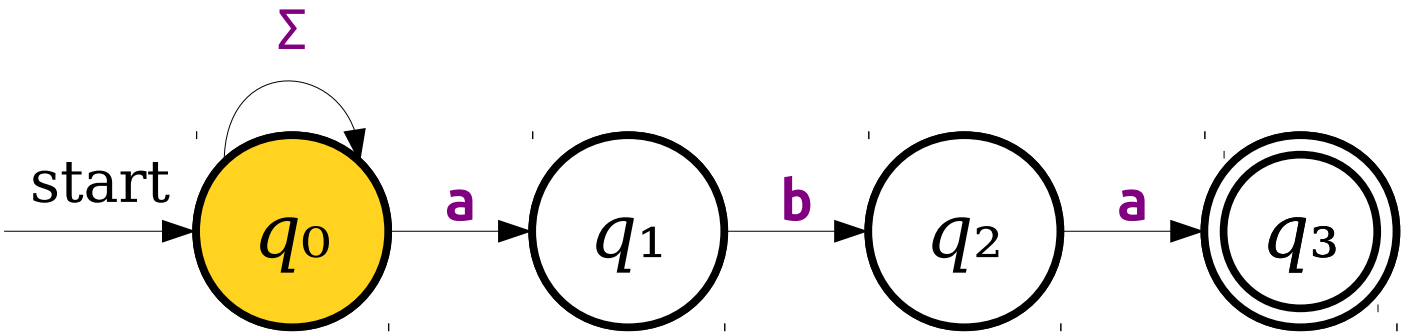


| | $a$ | $b$ |
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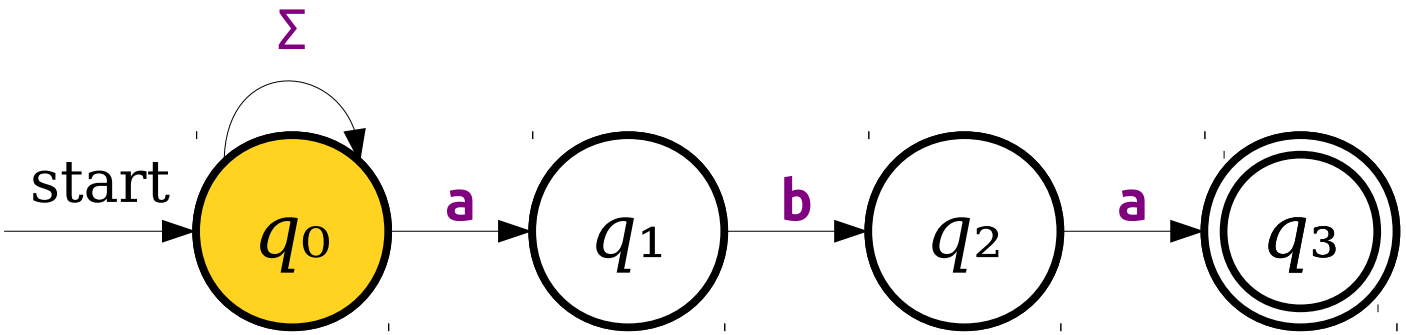


| | $a$ | $b$ |
|---|---|---|
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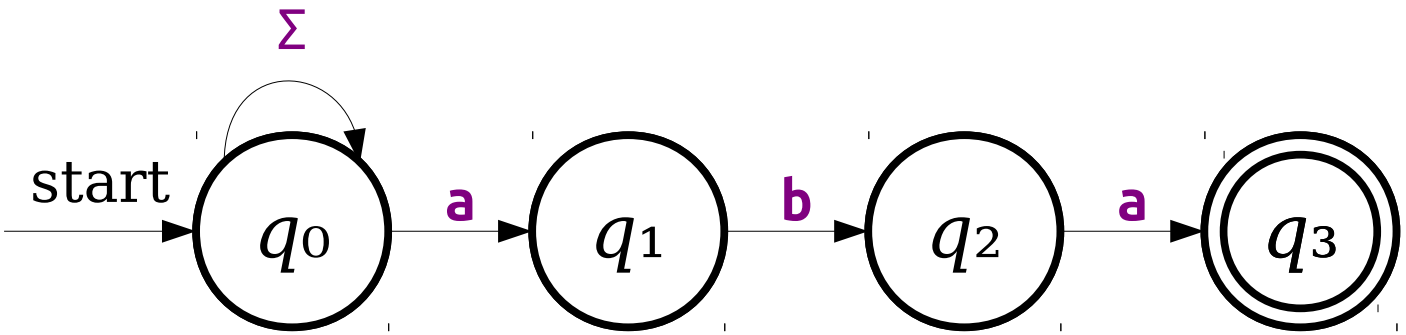


| | $a$ | $b$ |
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| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
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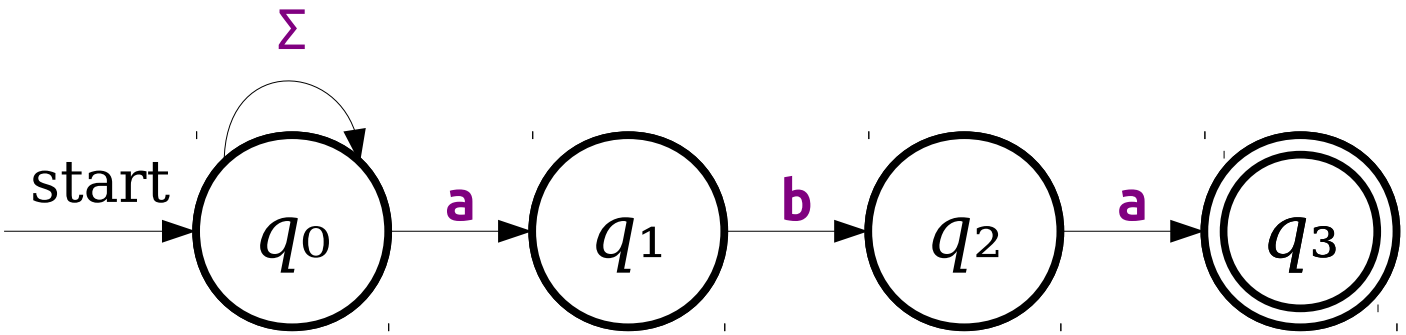


| | $a$ | $b$ |
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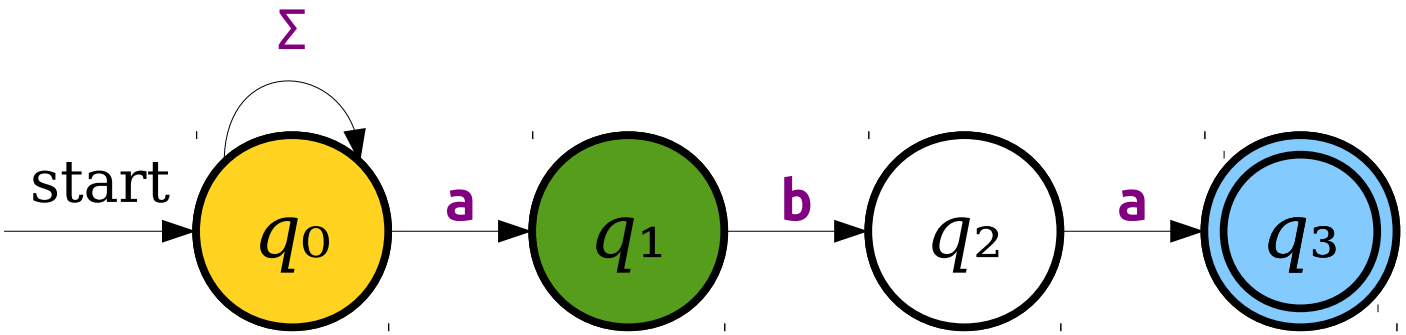
| $\{q_0, q_1, q_3\}$ | | |




| | $a$ | $b$ |
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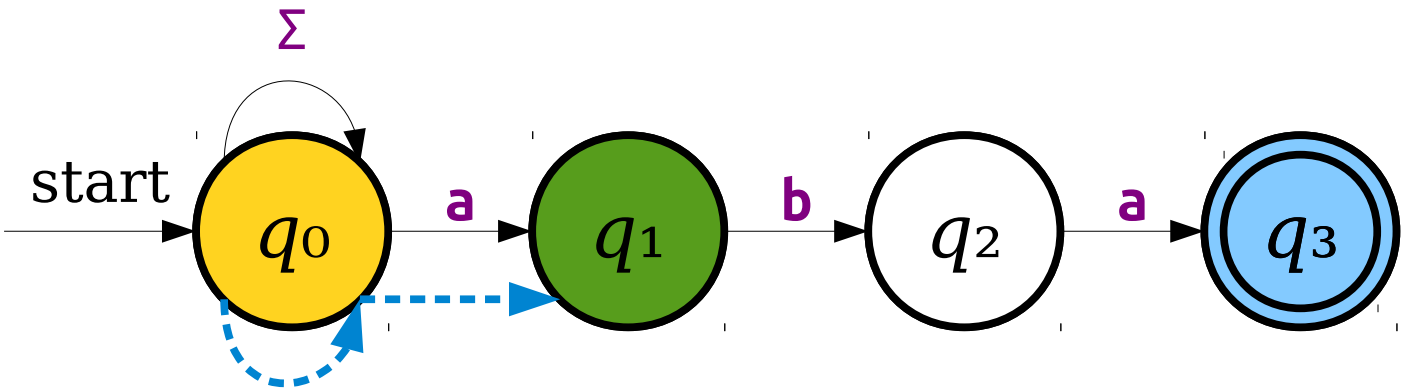


| | $a$ | $b$ |
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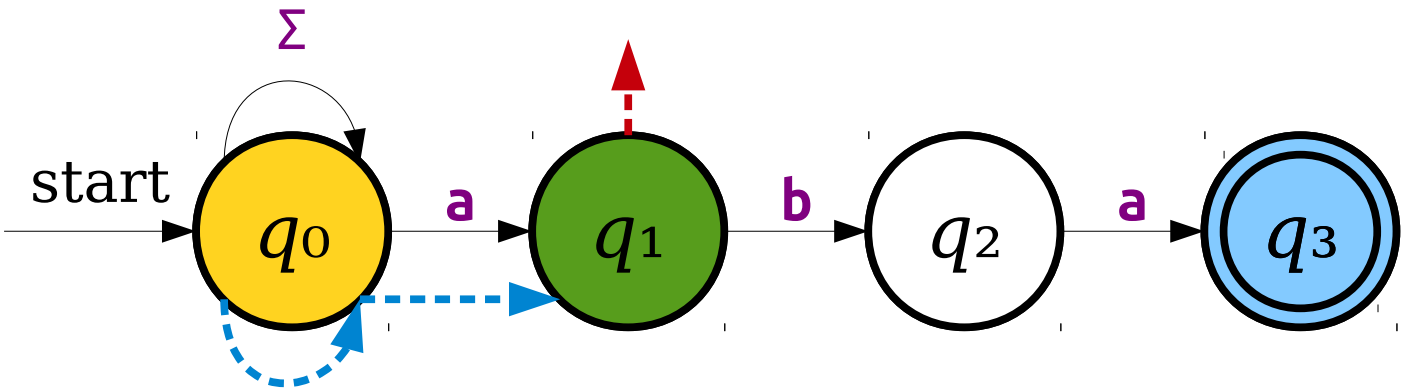
| $\{q_0, q_1, q_3\}$ | | |




| | $a$ | $b$ |
|---|---|---|
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| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
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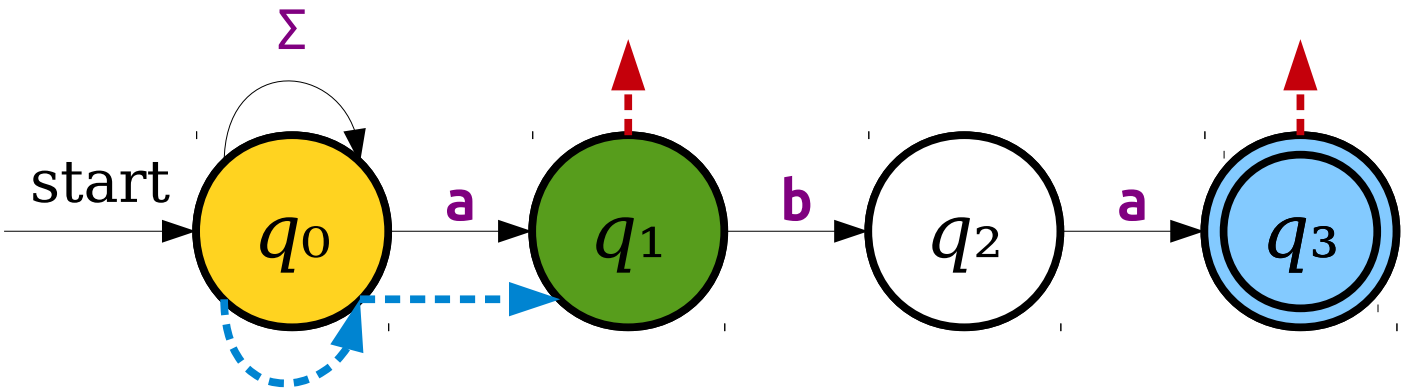
| $\{q_0, q_1, q_3\}$ | | |




| | $a$ | $b$ |
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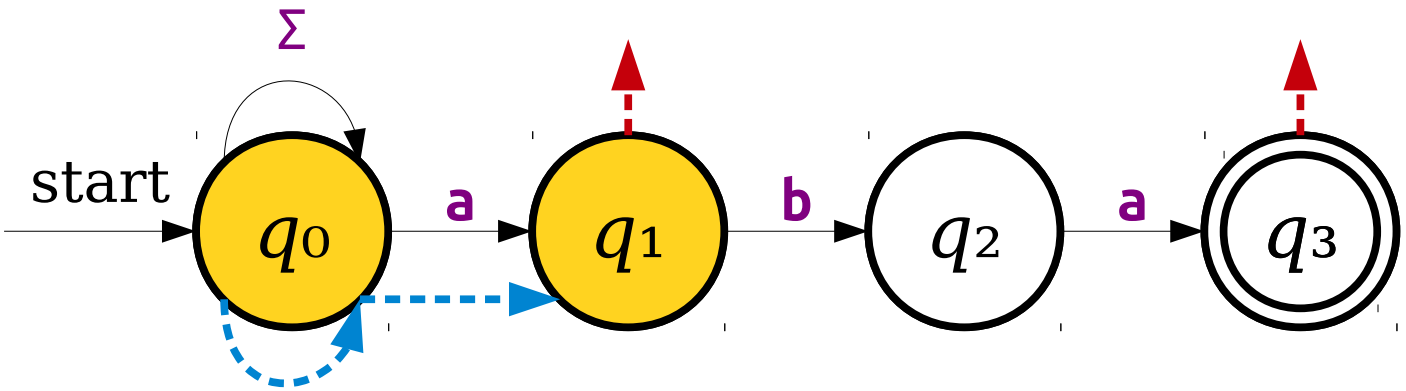
| $\{q_0, q_1, q_3\}$ | | |




| | $a$ | $b$ |
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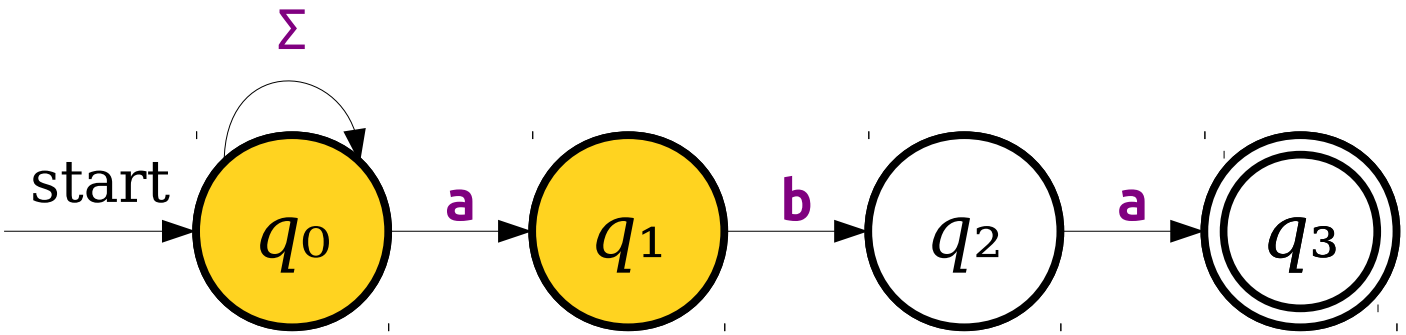
| $\{q_0, q_1, q_3\}$ | | |

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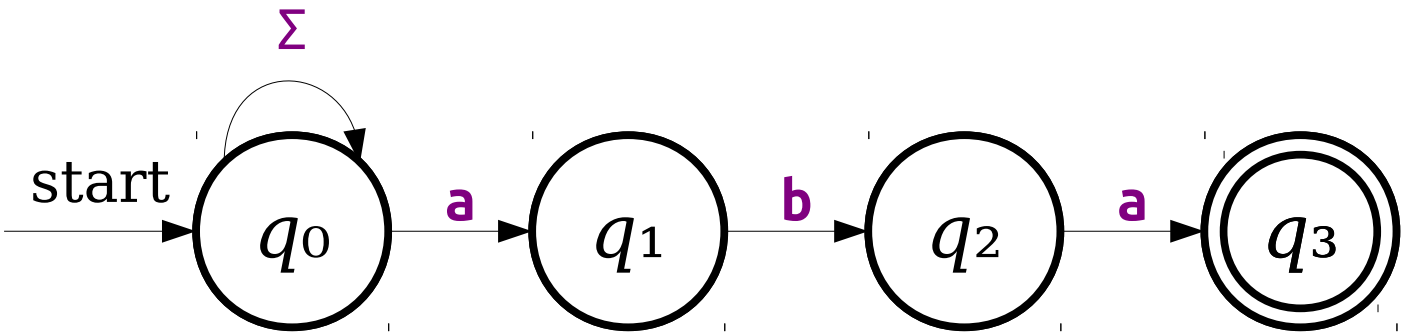


| | $a$ | $b$ |
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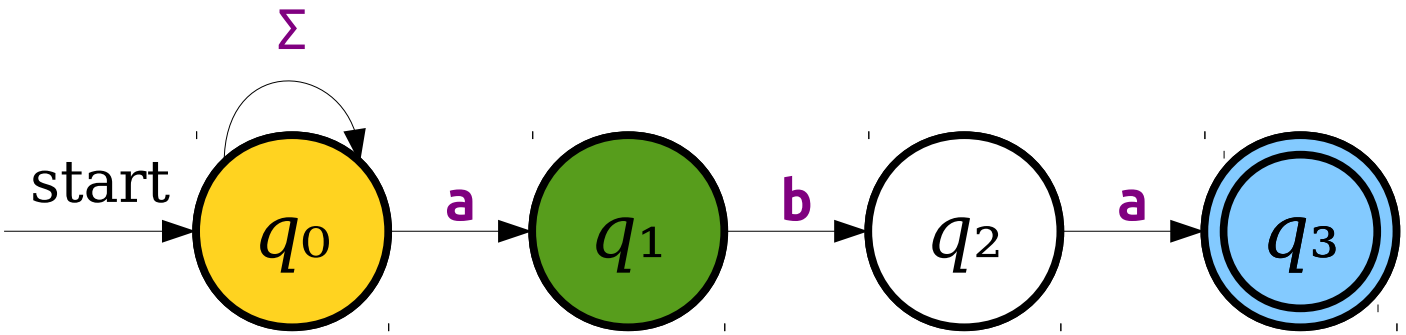


| | $a$ | $b$ |
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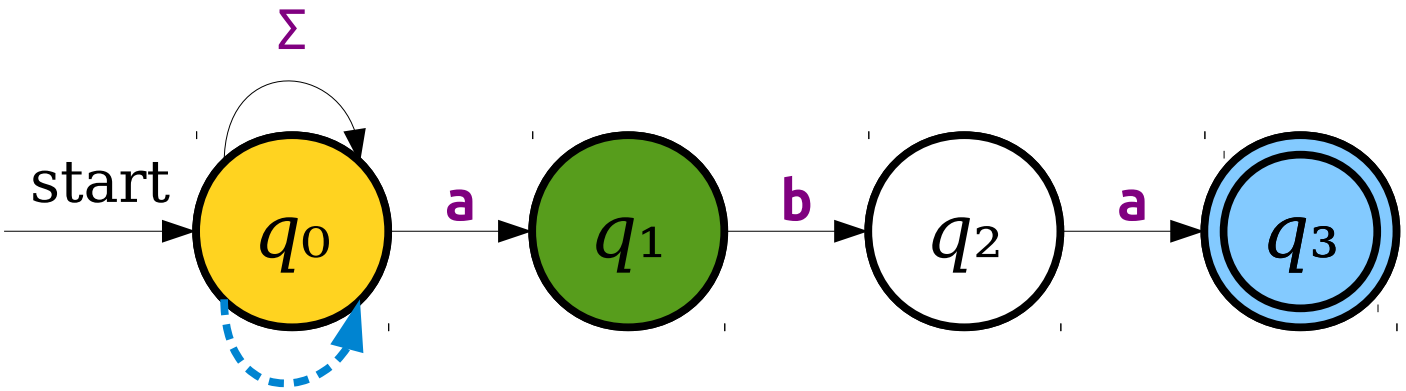


| | $a$ | $b$ |
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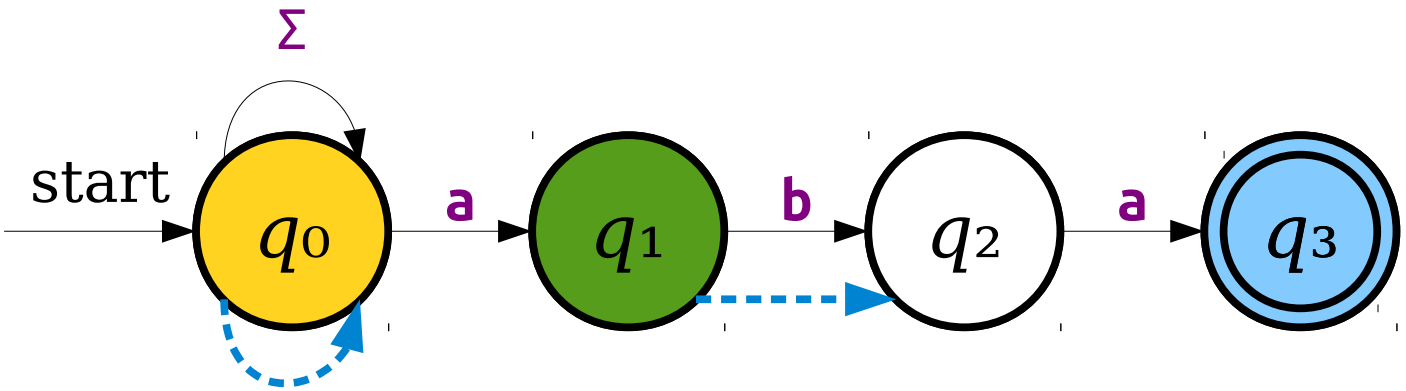


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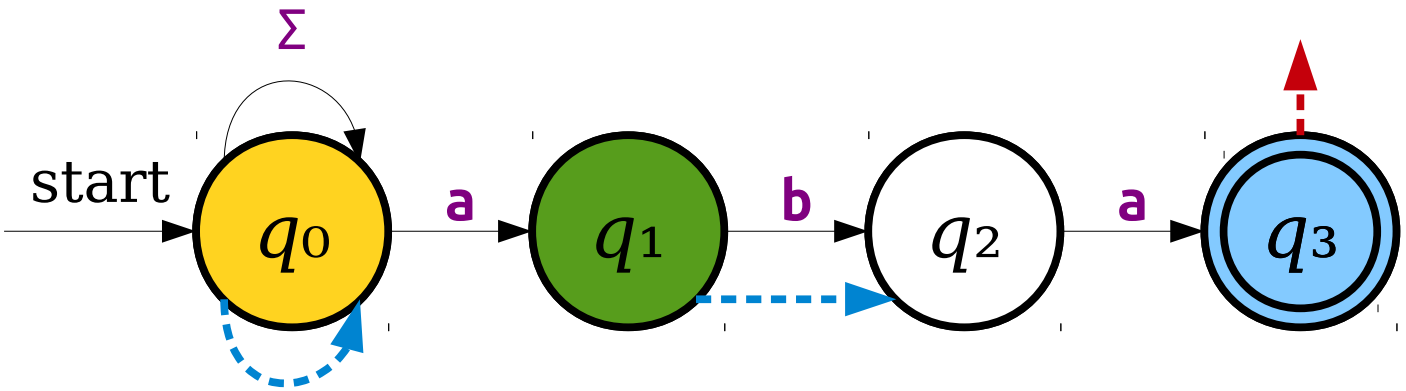


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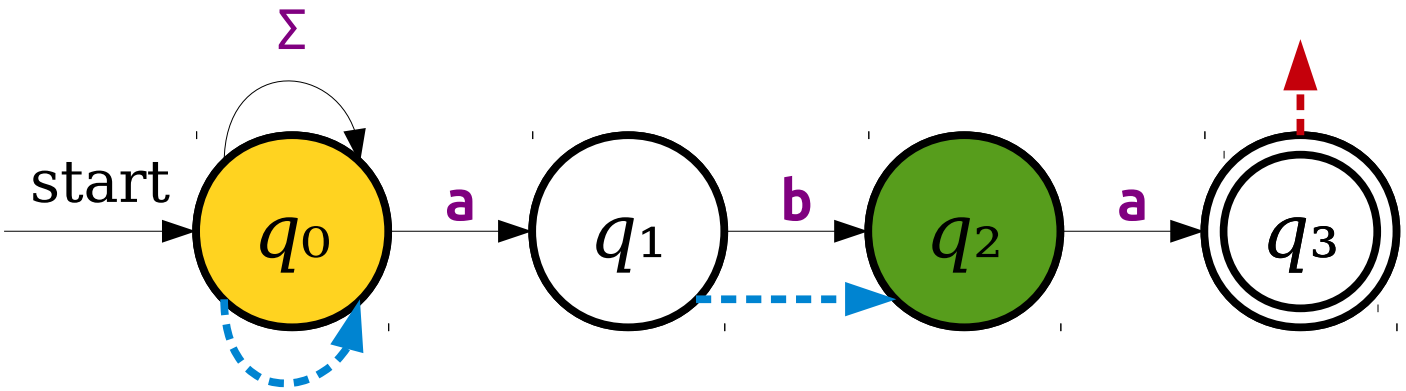


| | $a$ | $b$ |
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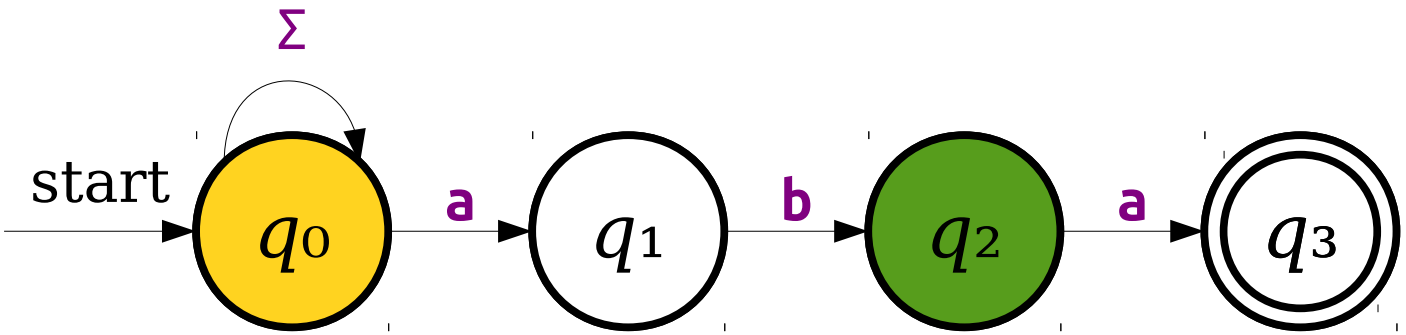


| | $a$ | $b$ |
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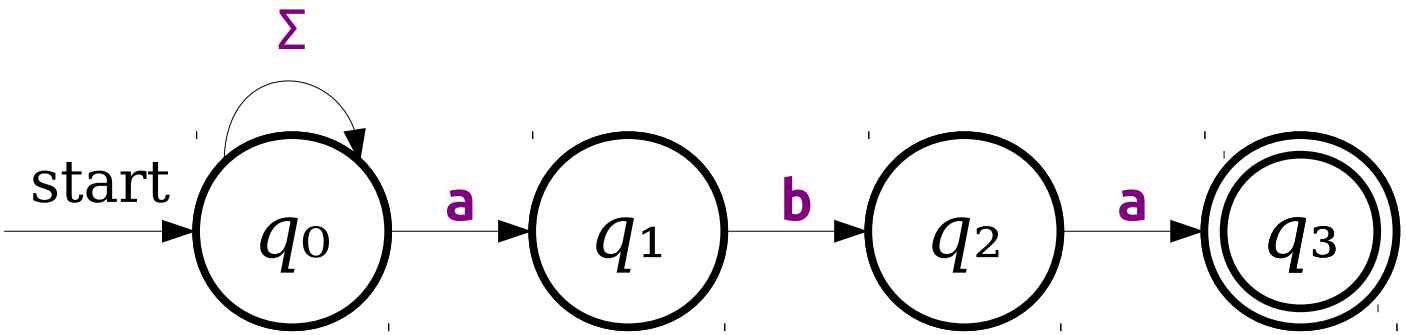


| | $a$ | $b$ |
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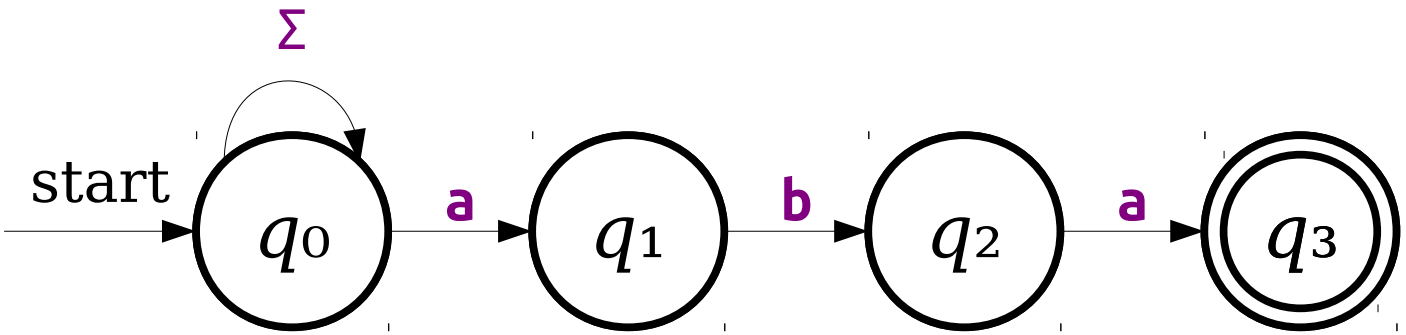


| | $a$ | $b$ |
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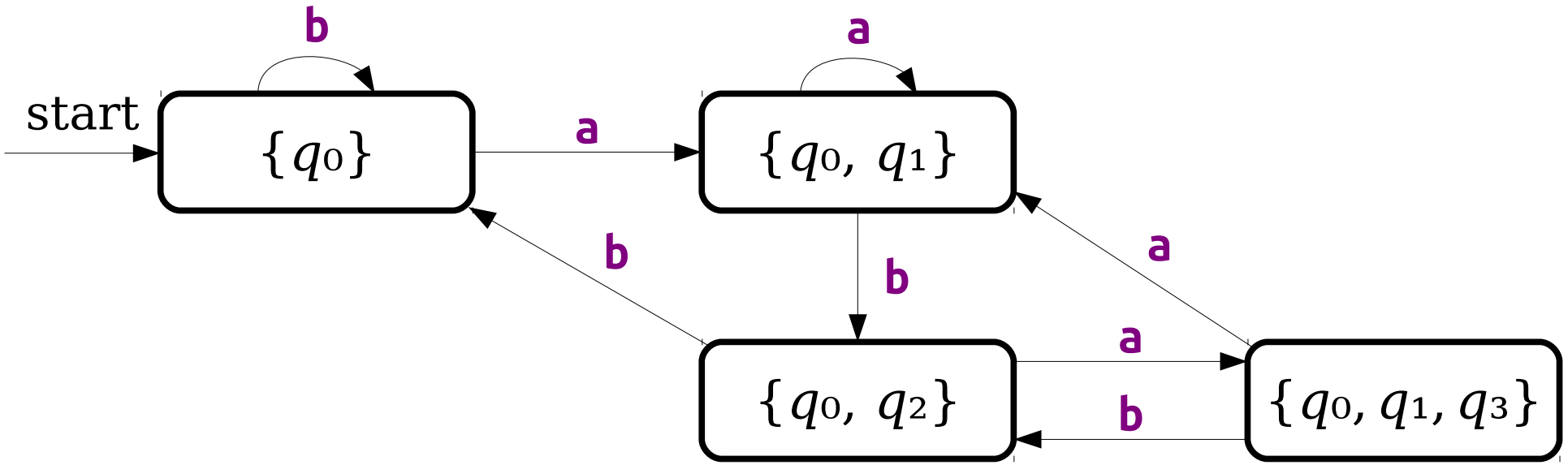

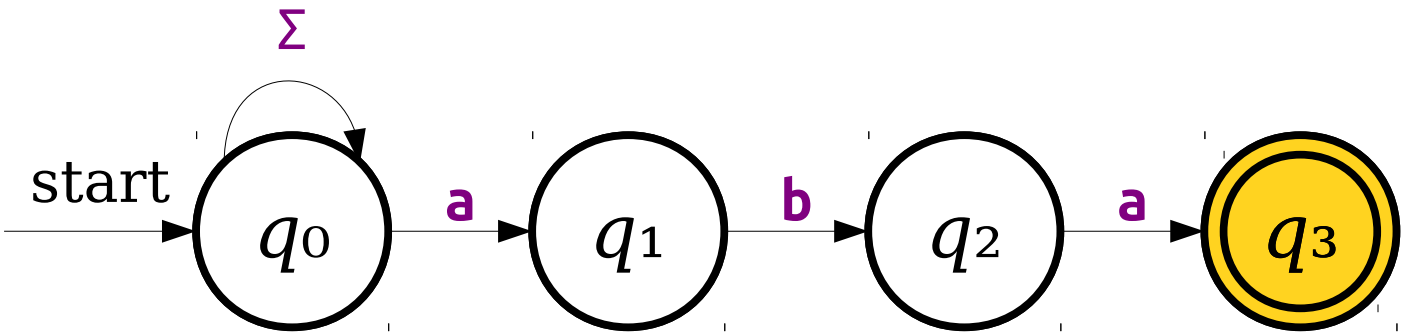


| | $a$ | $b$ |
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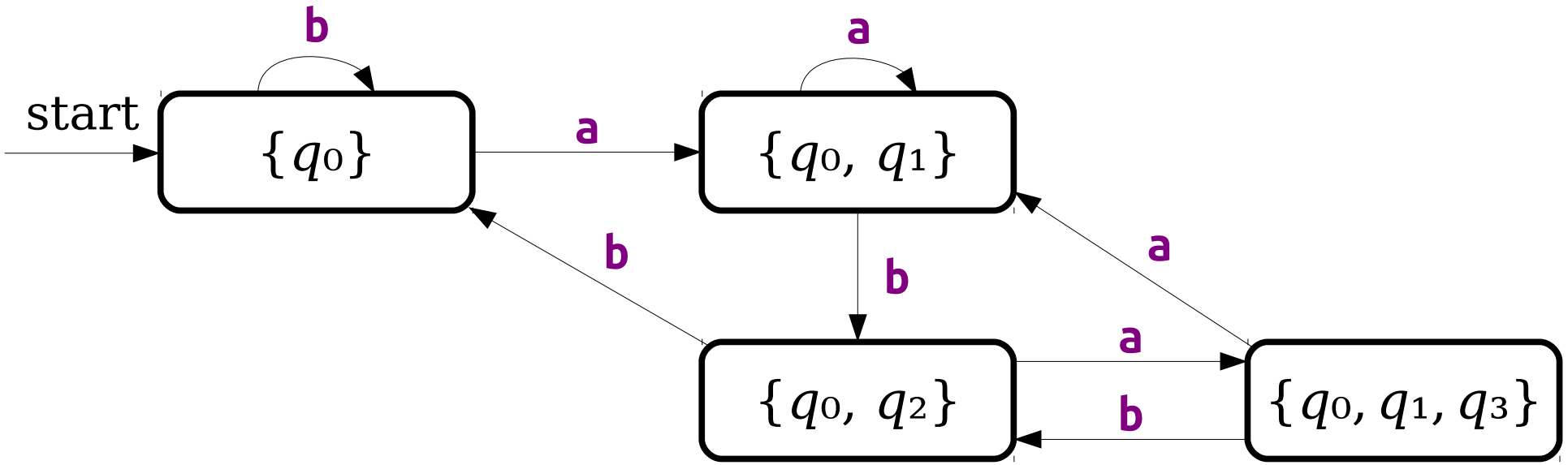

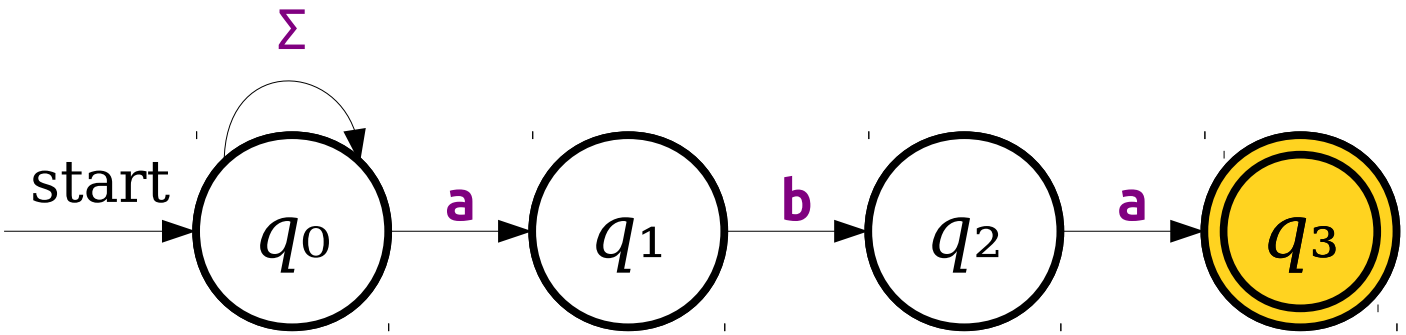


| | $a$ | $b$ |
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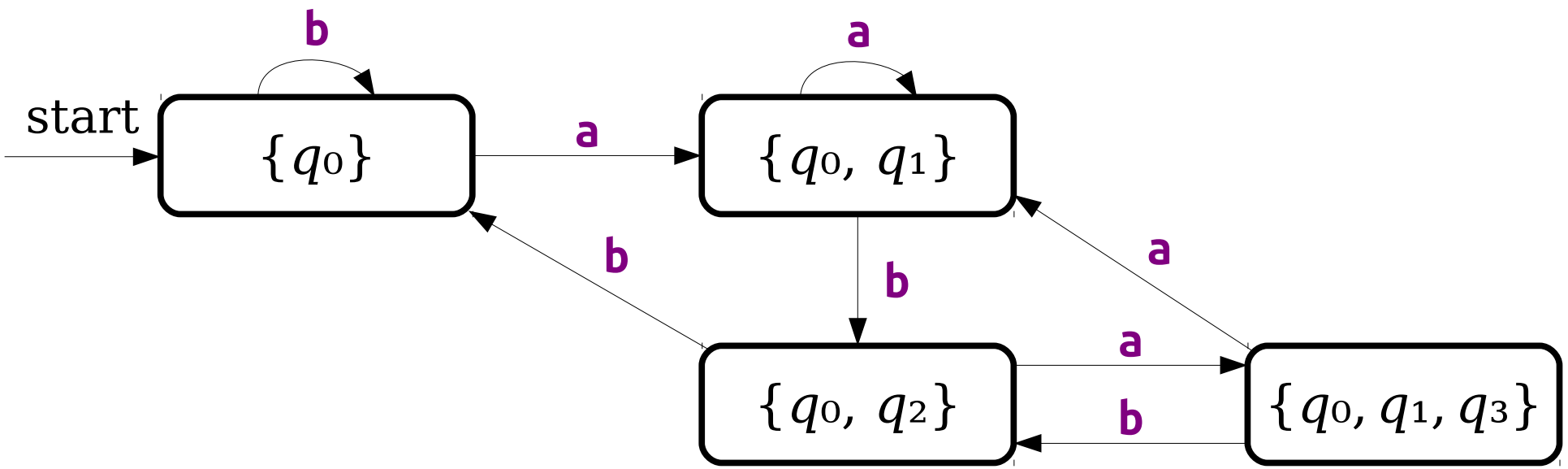

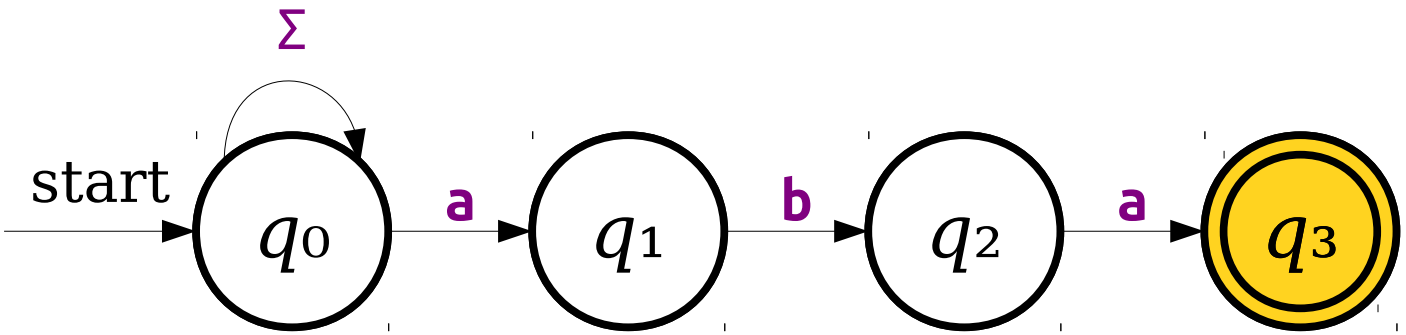


| | $a$ | $b$ |
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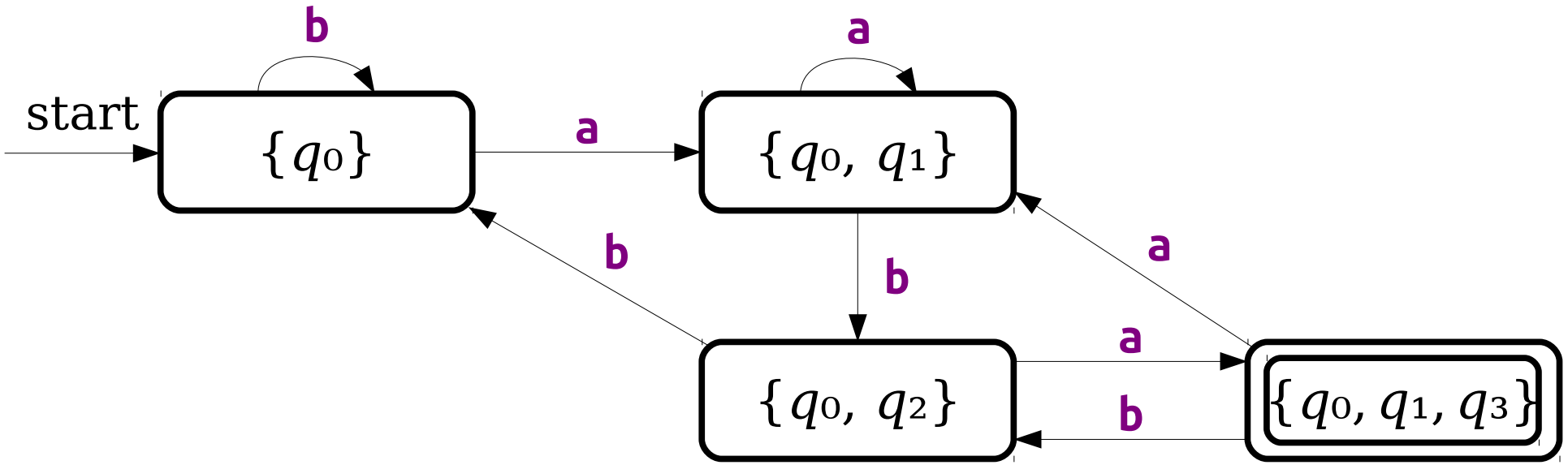

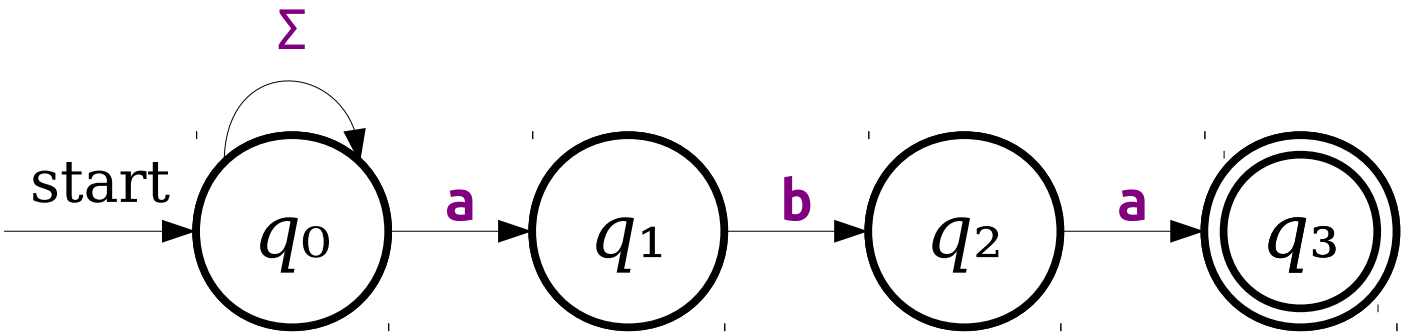


| | $a$ | $b$ |
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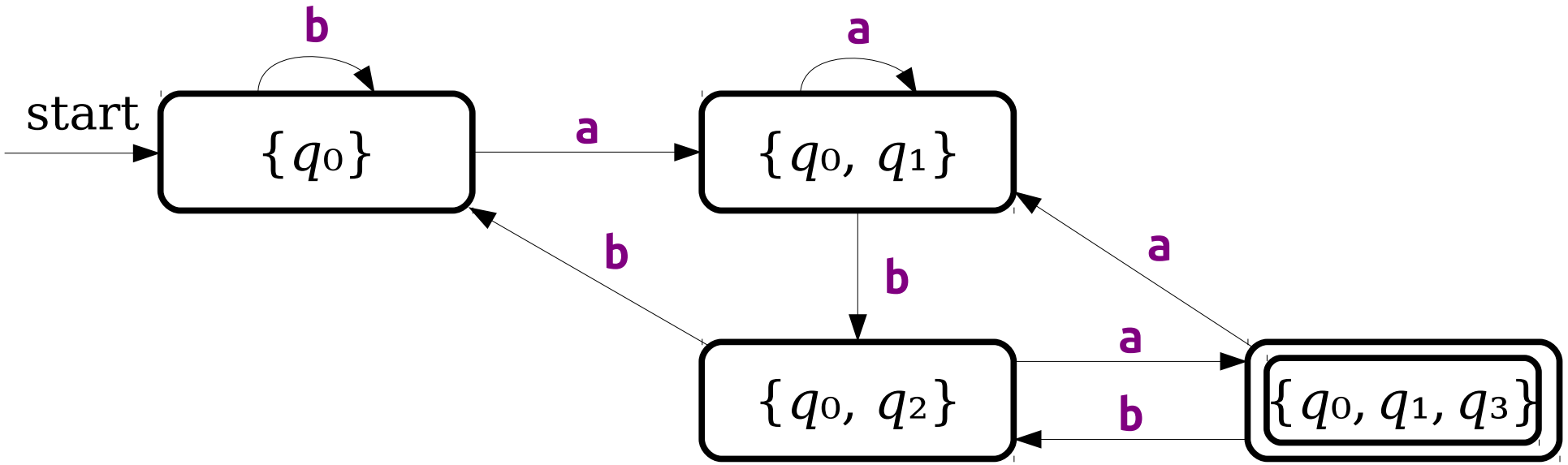

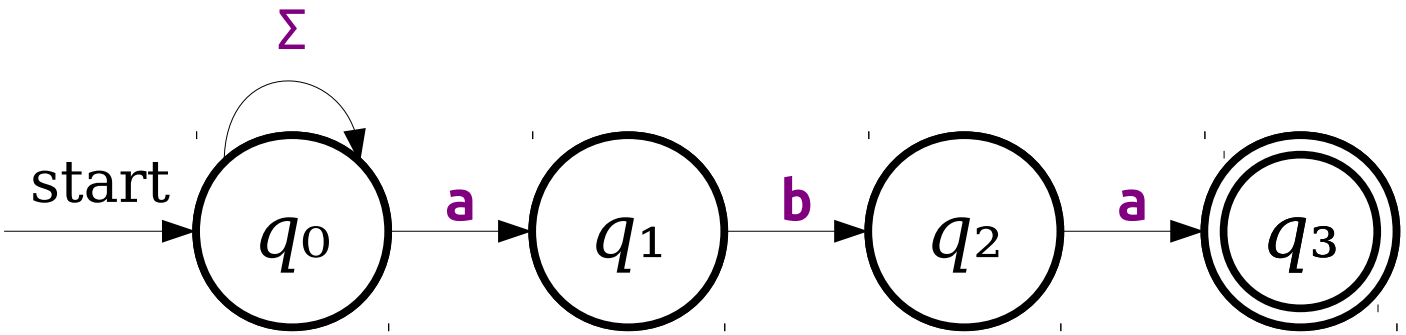


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|---|---|---|---|---|---|
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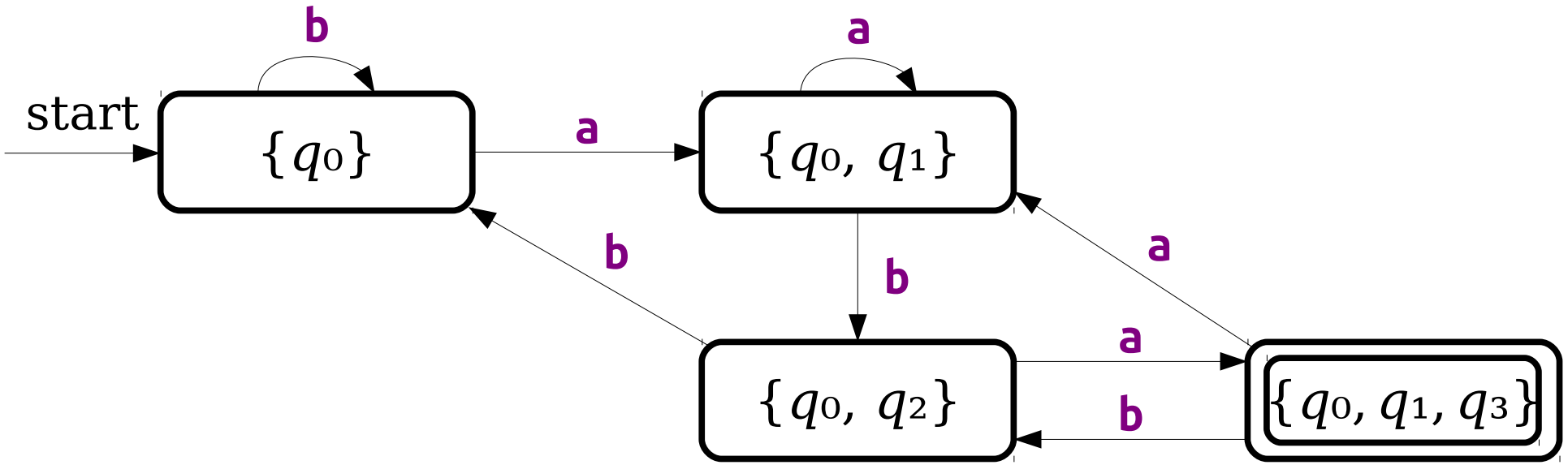

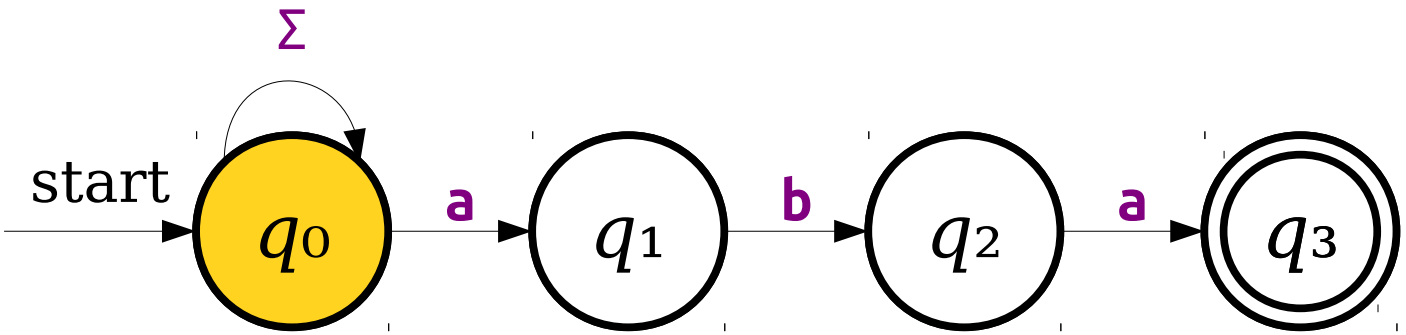


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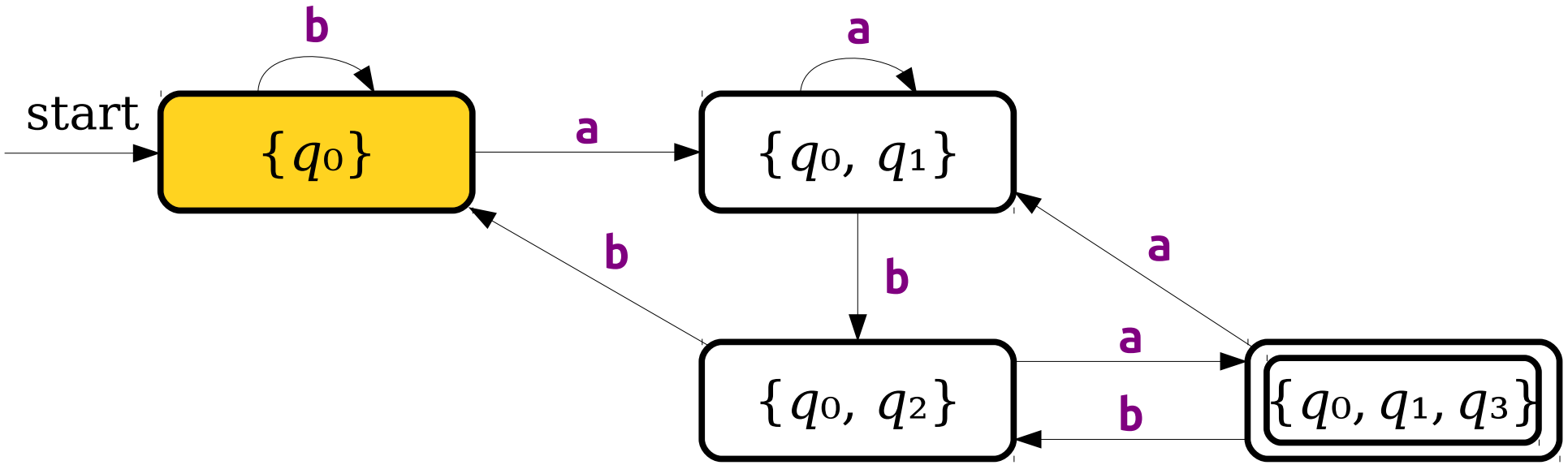

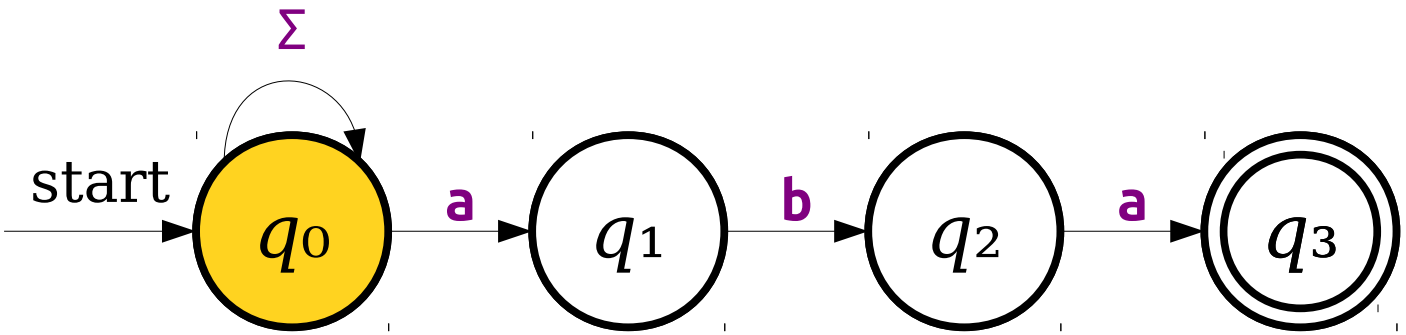
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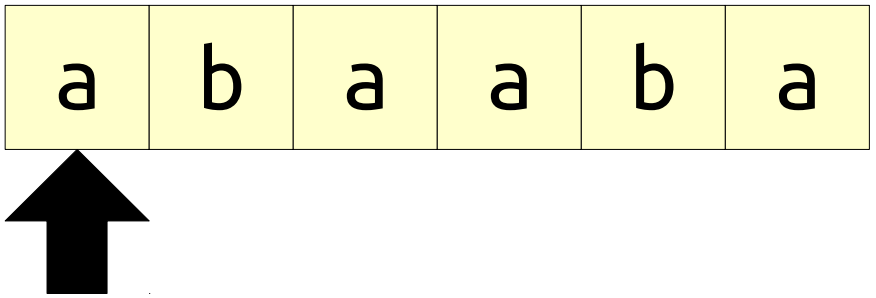
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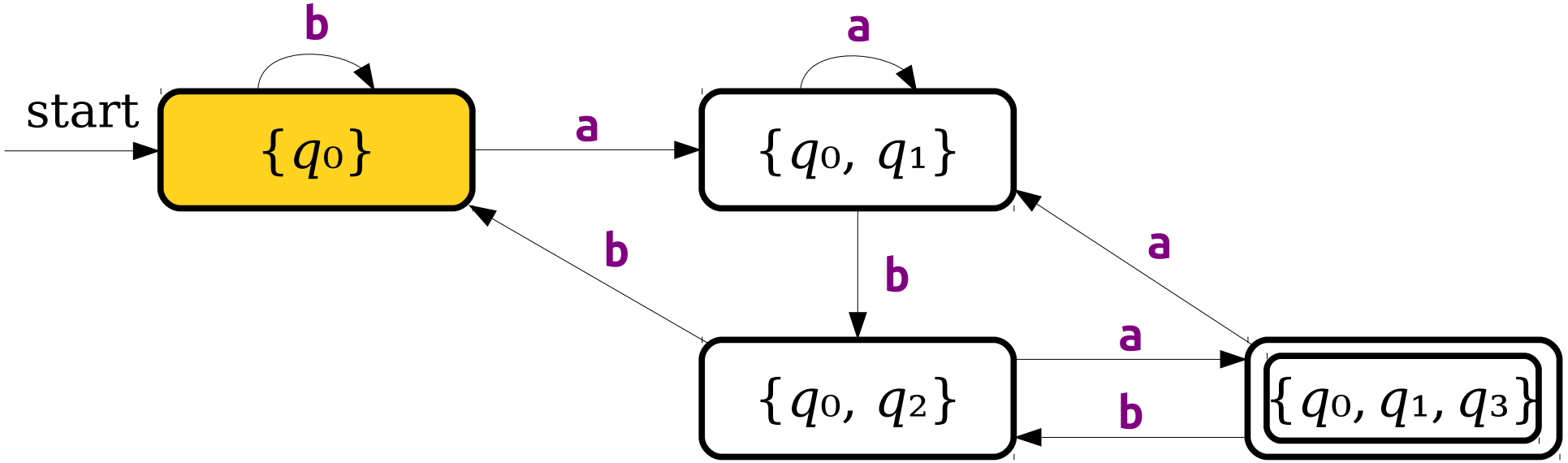
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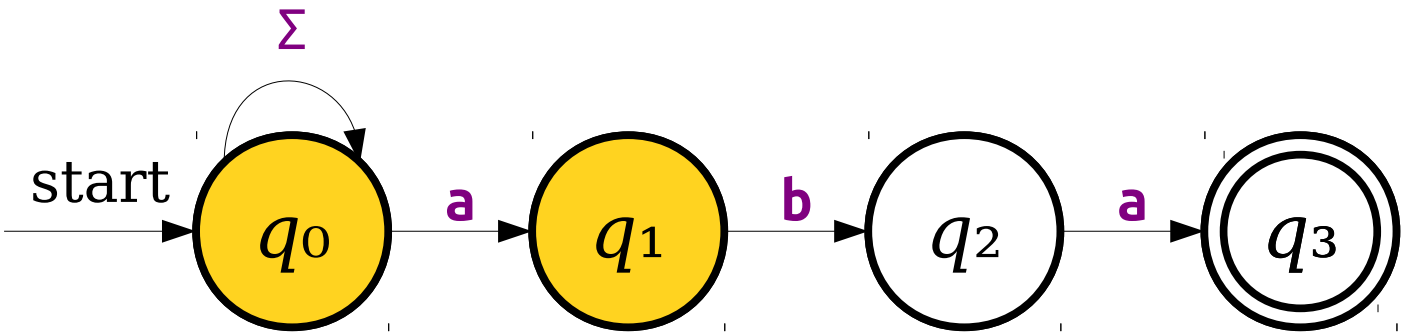
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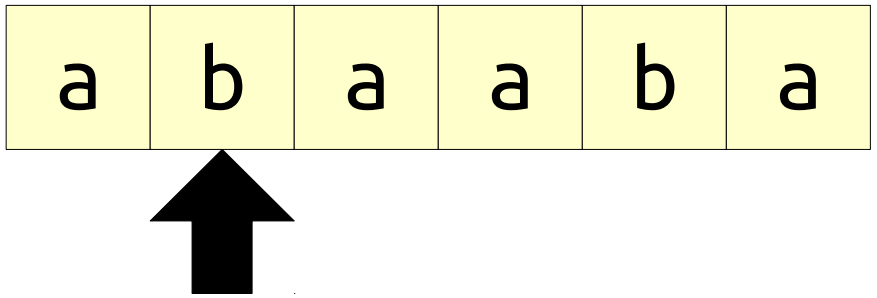
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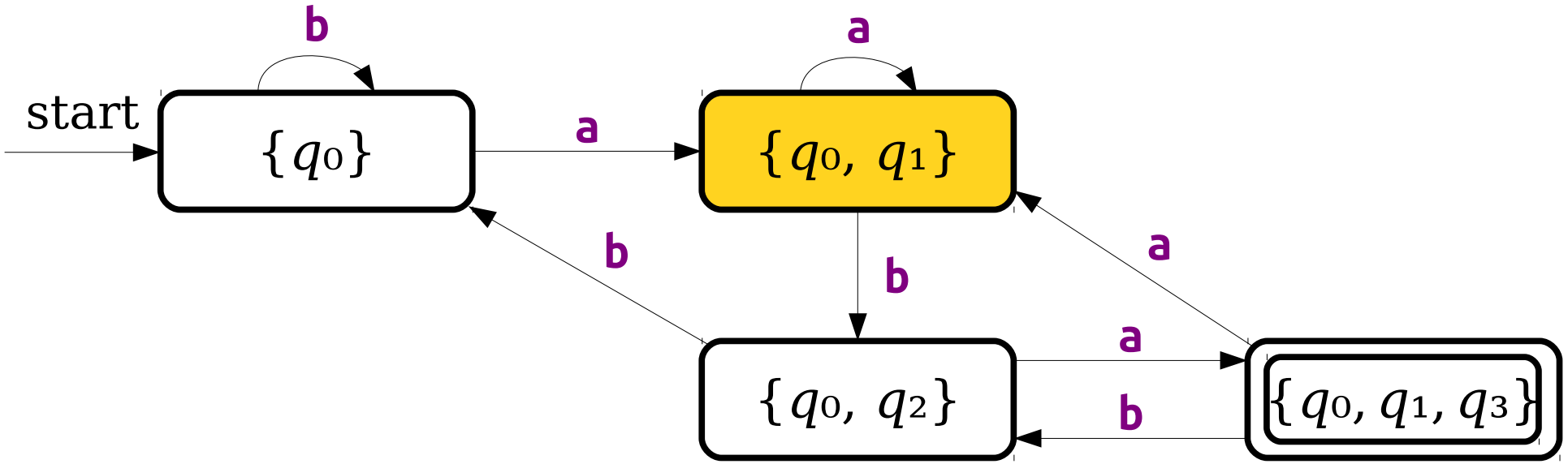
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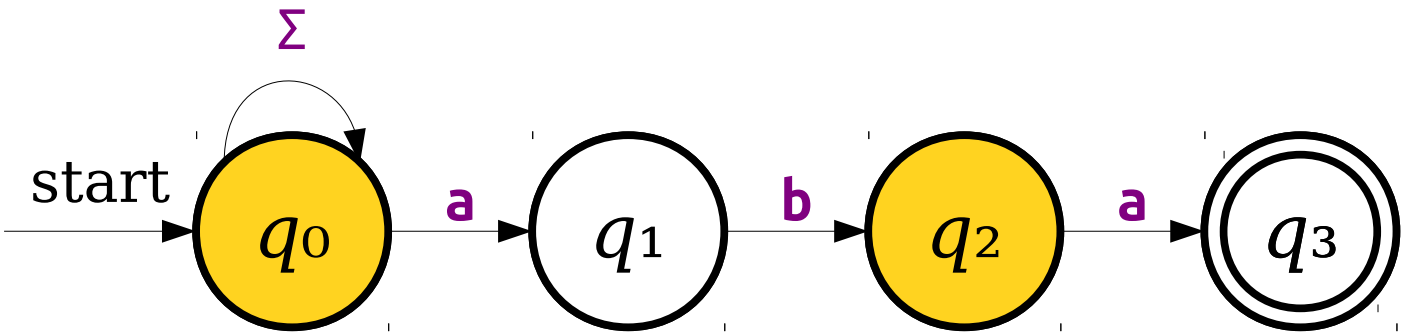
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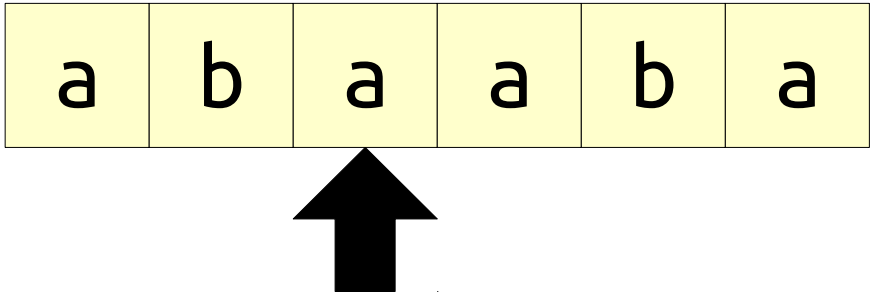
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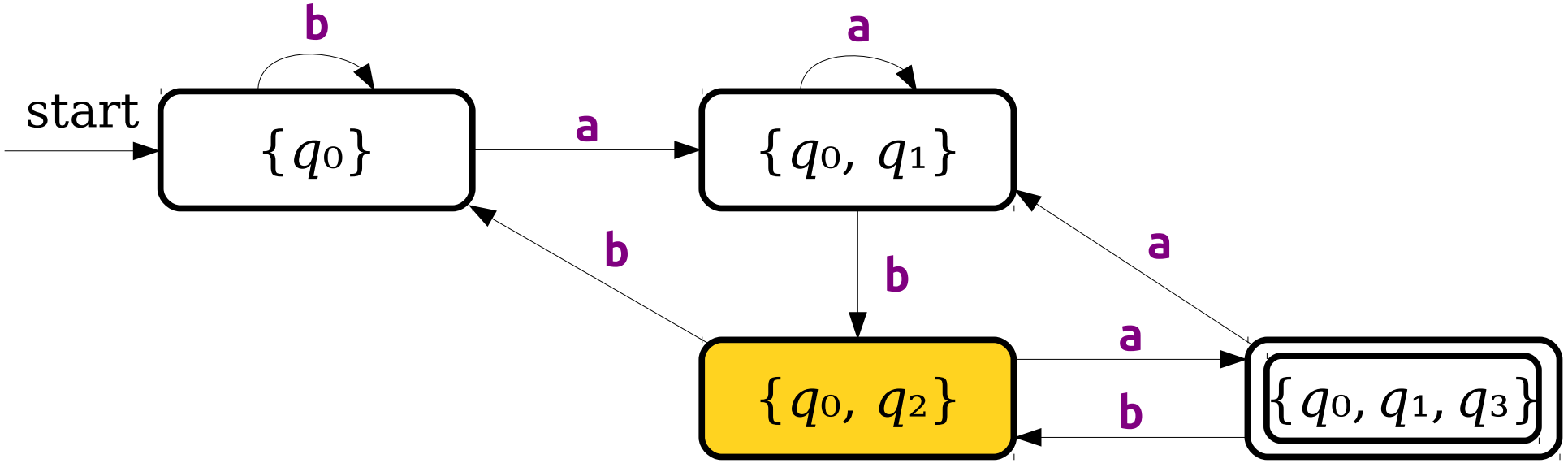
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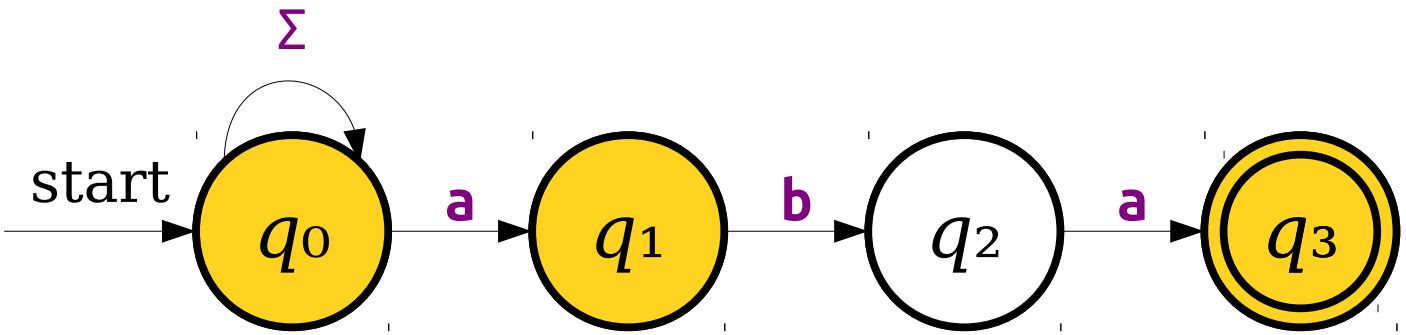
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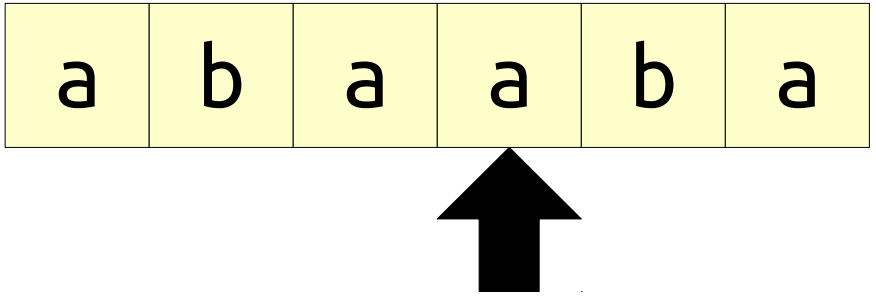
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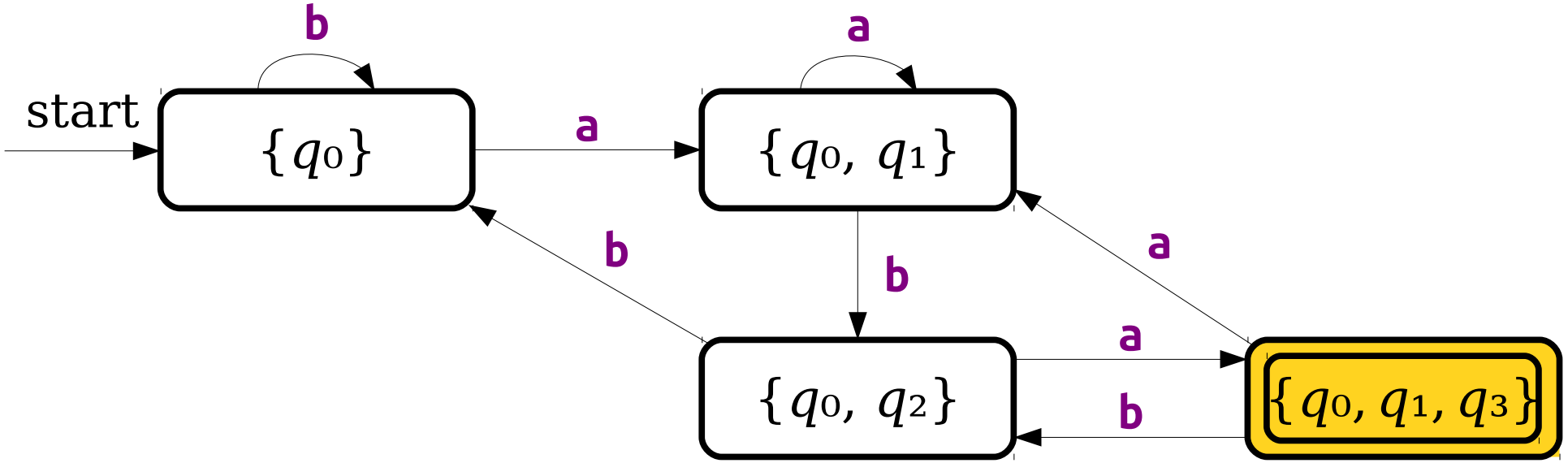
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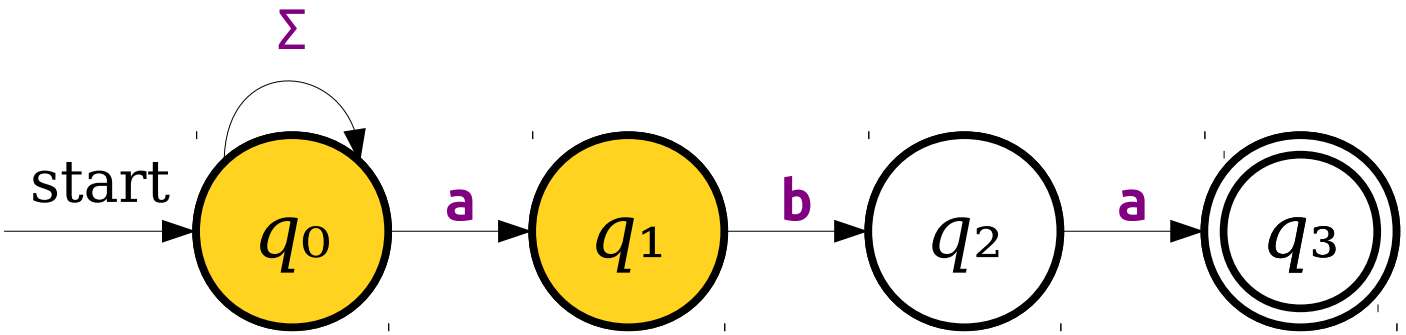
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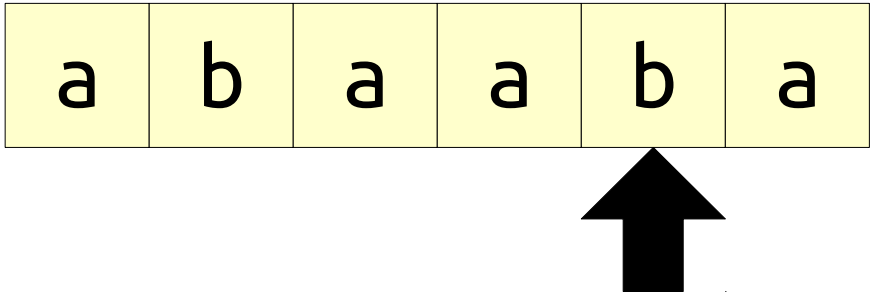
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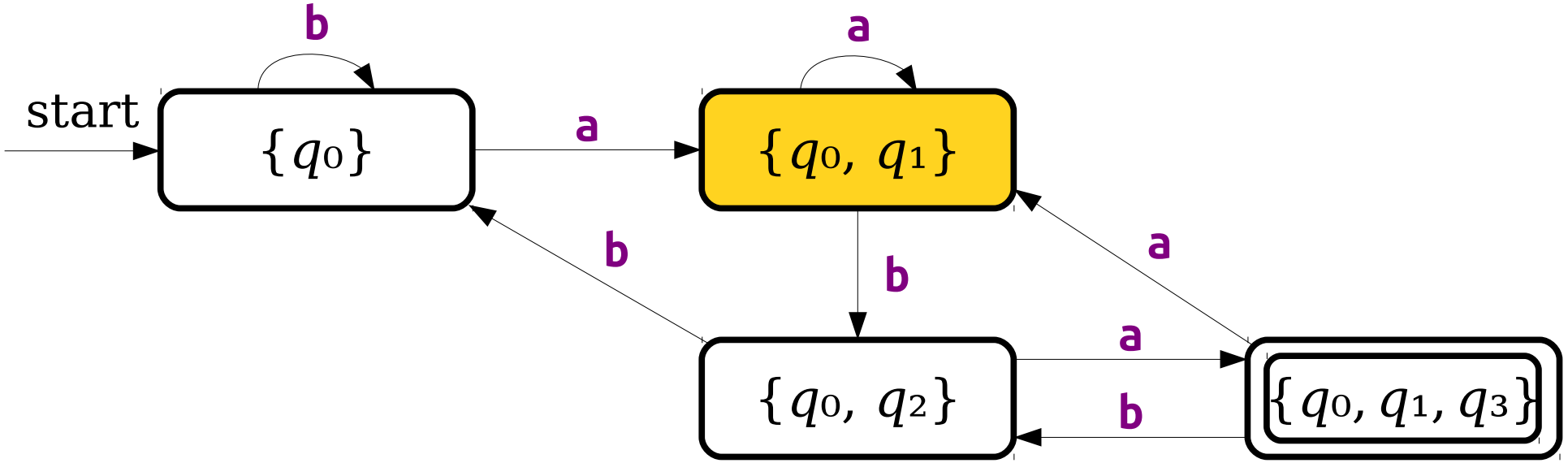
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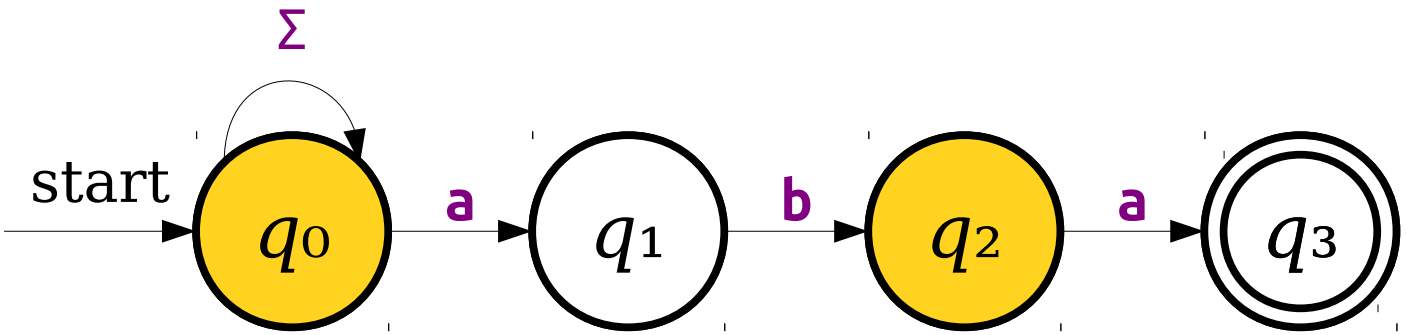
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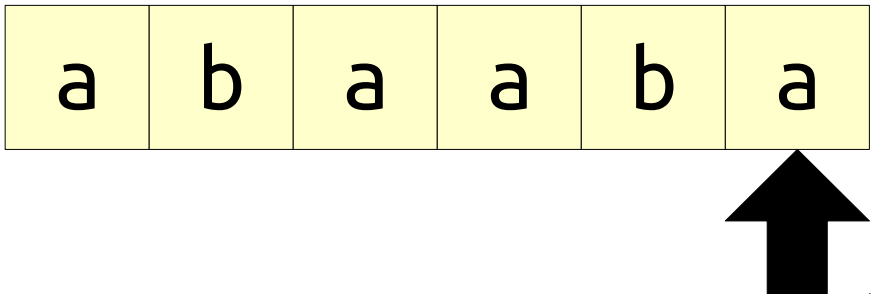
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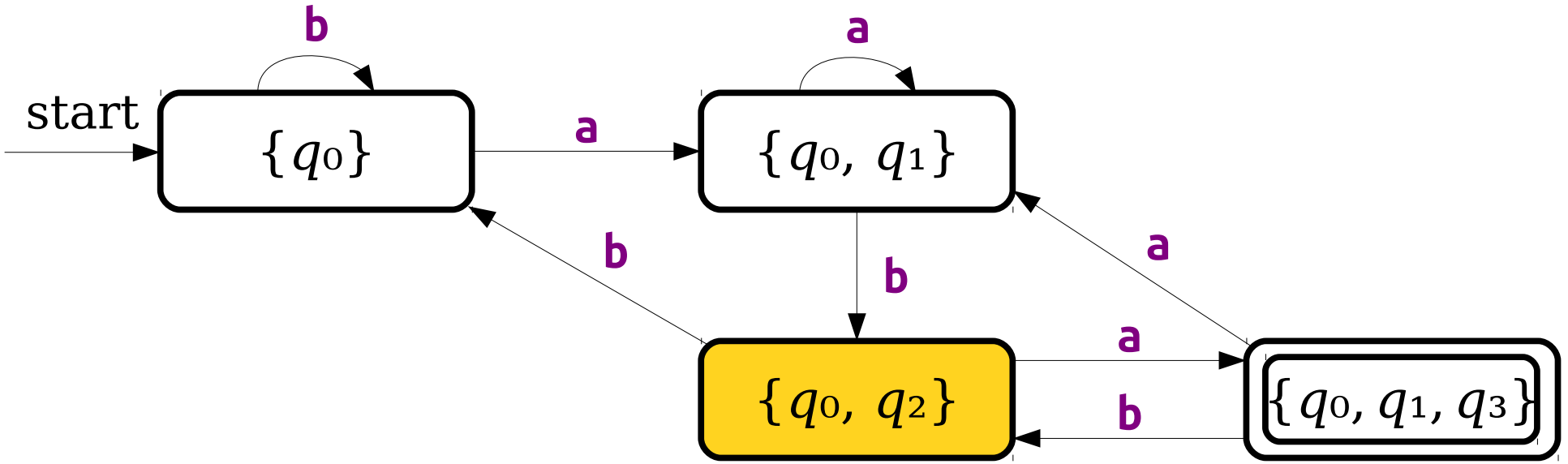
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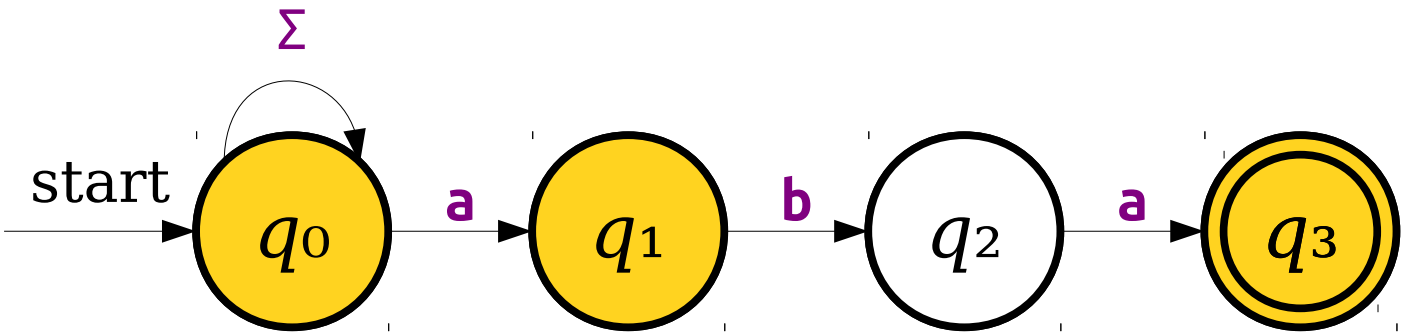


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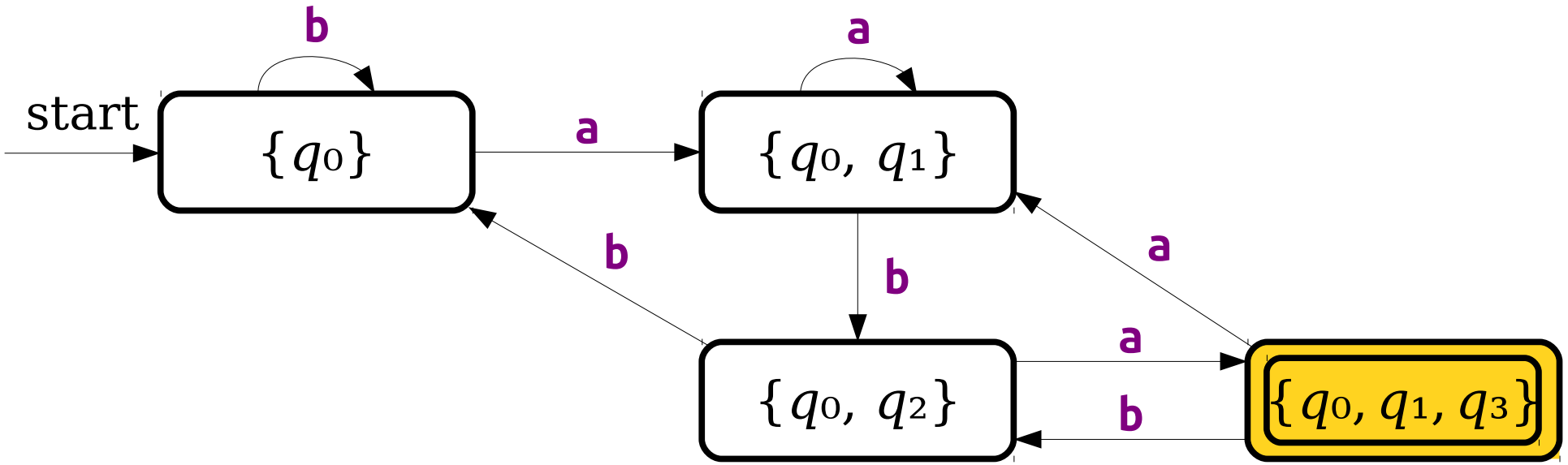

# The Subset Construction

- This procedure for turning an NFA for a language $L$ into a DFA for a language $L$ is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.

# The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set $S$.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size $n$, but no DFAs of size less than $2^n$?

A language $L$ is called a ***regular language*** if there exists a DFA $D$ such that $\mathcal{L}(D) = L$.

# An Important Result

***Theorem:*** A language $L$ is regular if and only if there is some NFA $N$ such that $\mathcal{L}(N) = L$.

***Proof Sketch:*** Pick a language $L$. First, assume $L$ is regular. That means there's a DFA $D$ where $\mathcal{L}(D) = L$. Every DFA is "basically" an NFA, so there's an NFA ($D$) whose language is $L$.

Next, assume there's an NFA $N$ such that $\mathcal{L}(N) = L$. Using the subset construction, we can build a DFA $D$ where $\mathcal{L}(N) = \mathcal{L}(D)$. Then we have that $\mathcal{L}(D) = L$, so $L$ is regular. ■-*ish*

# Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

# Properties of Regular Languages

# The Complement of a Language

- Given a language $L \subseteq \Sigma^*$, the ***complement*** of that language (denoted $\overline{L}$) is the language of all strings in $\Sigma^*$ that aren't in $L$.
- Formally:

$$\overline{L} = \Sigma^* - L$$

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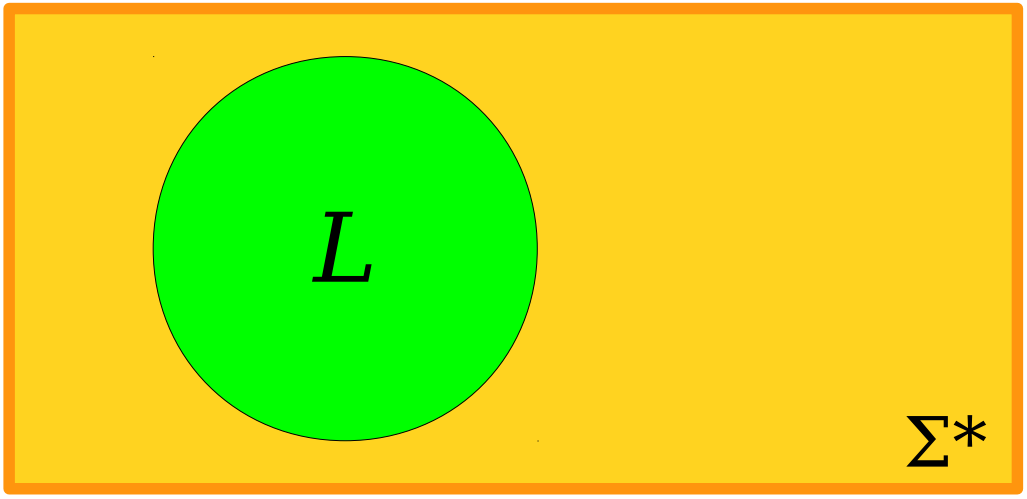

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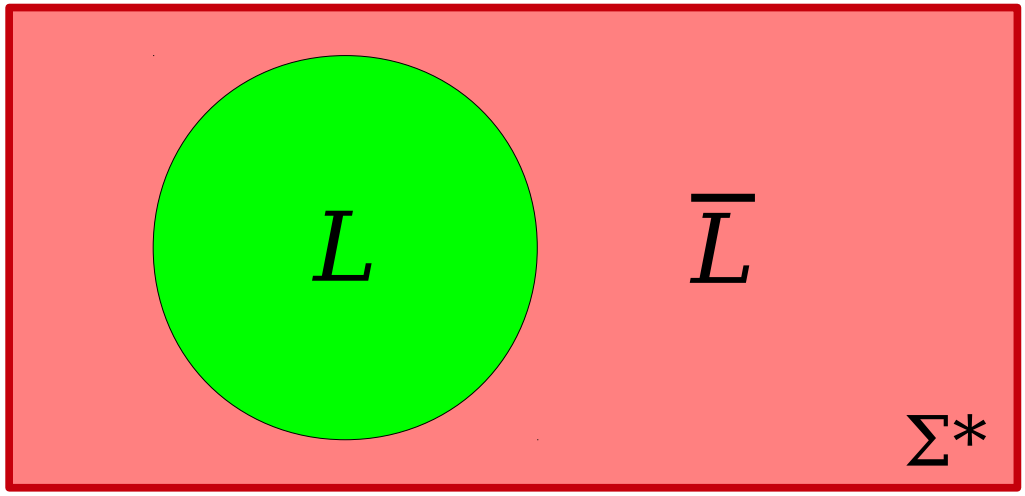

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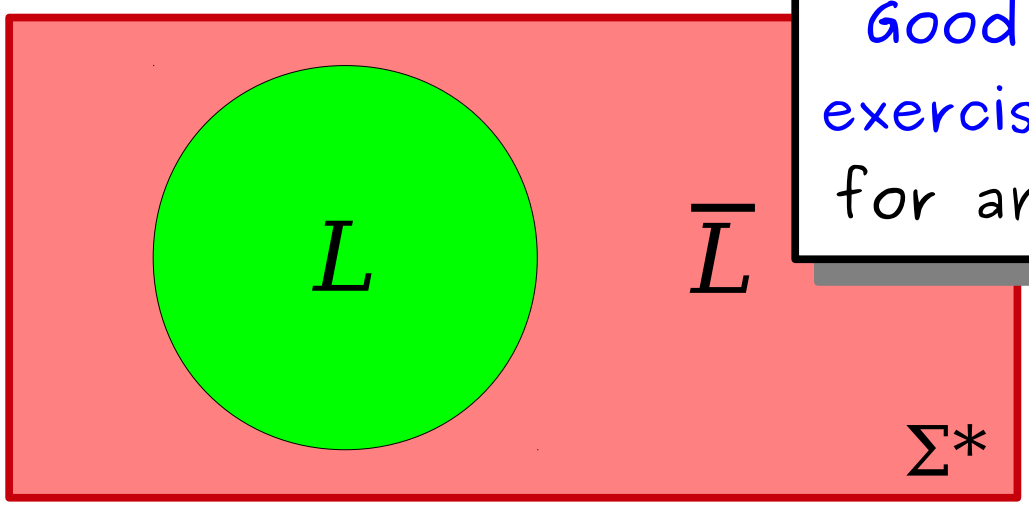
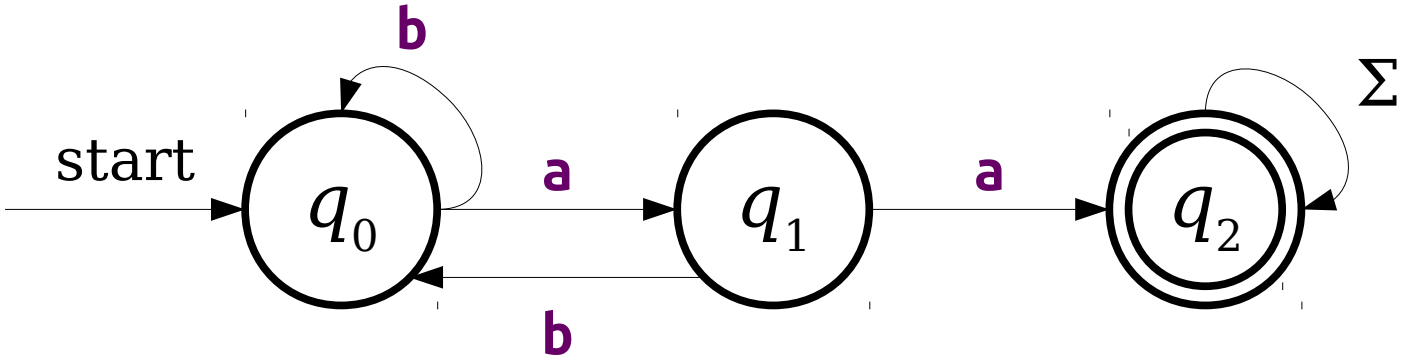


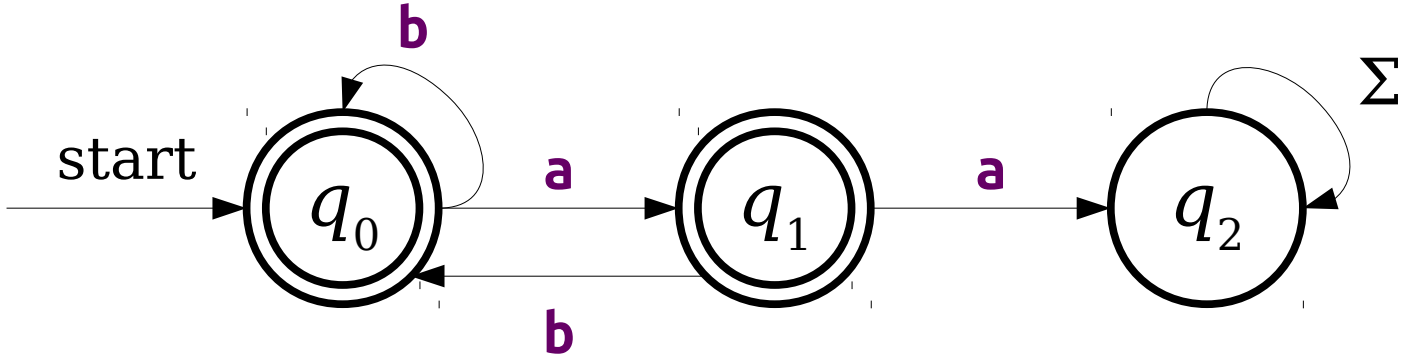

Good proofwriting exercise: prove $\overline{\overline{L}} = L$ for any language $L$.

# Complementing Regular Languages

$L = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w$ contains **aa** as a substring $\}$




$\overline{L} = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w$ ***does not*** contain **aa** as a substring $\}$



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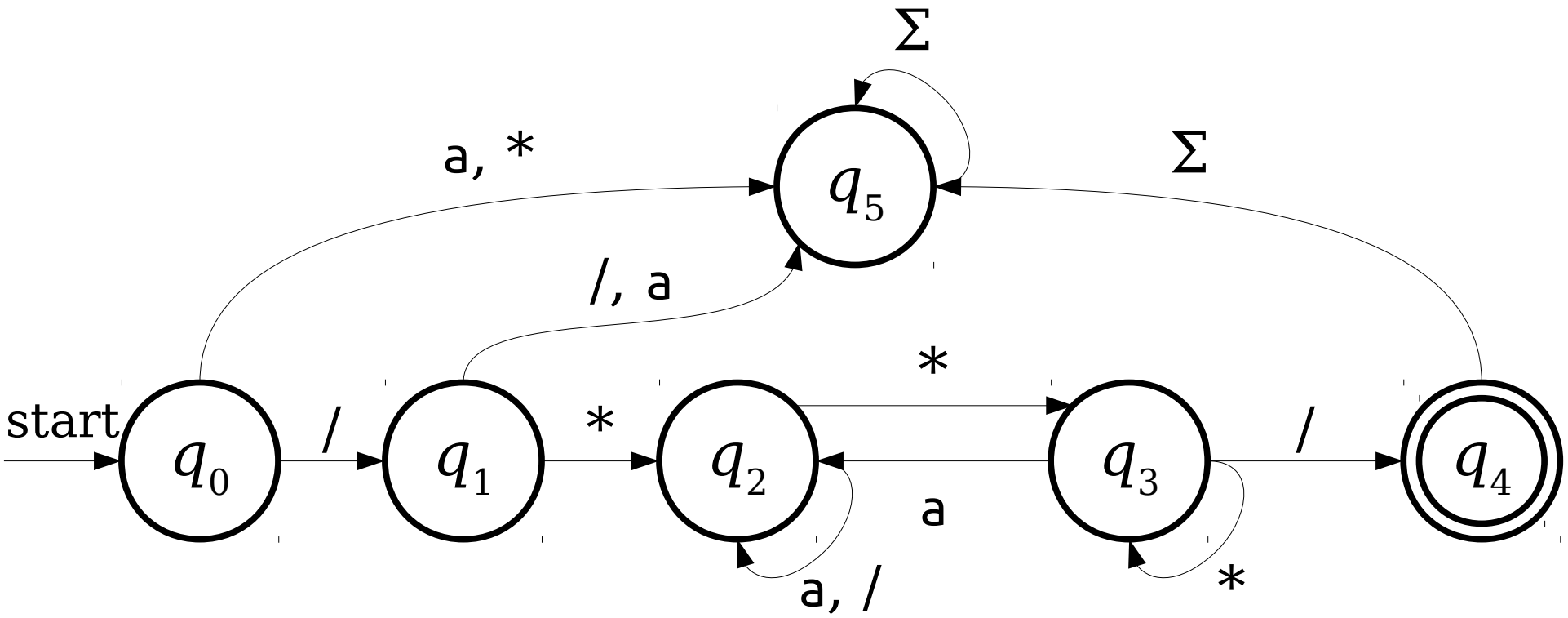
$L = \{\ w \in \{\mathtt{a}, \mathtt{*}, \mathtt{/}\}^* \mid w$ represents a C-style comment $\}$



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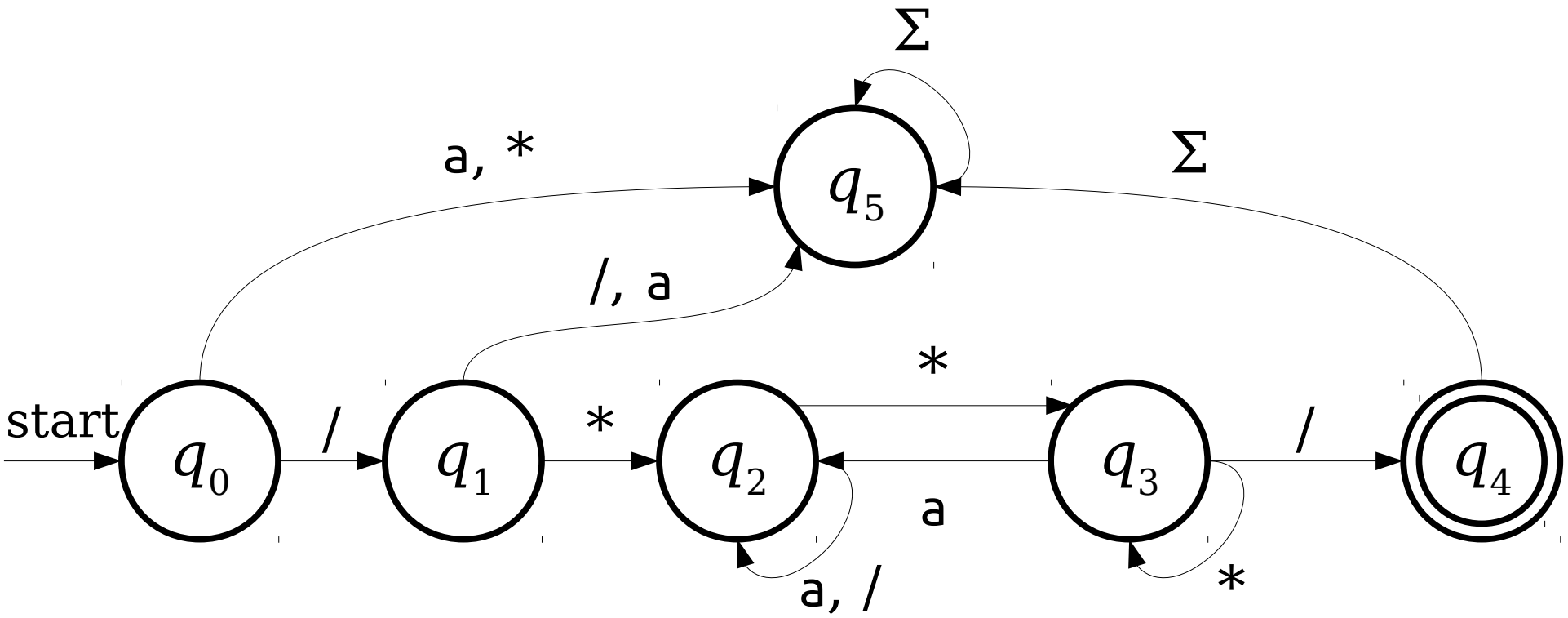
$\overline{L}$ = { $w \in$ {a, *, /}* | $w$ ***doesn't*** represent a C-style comment }



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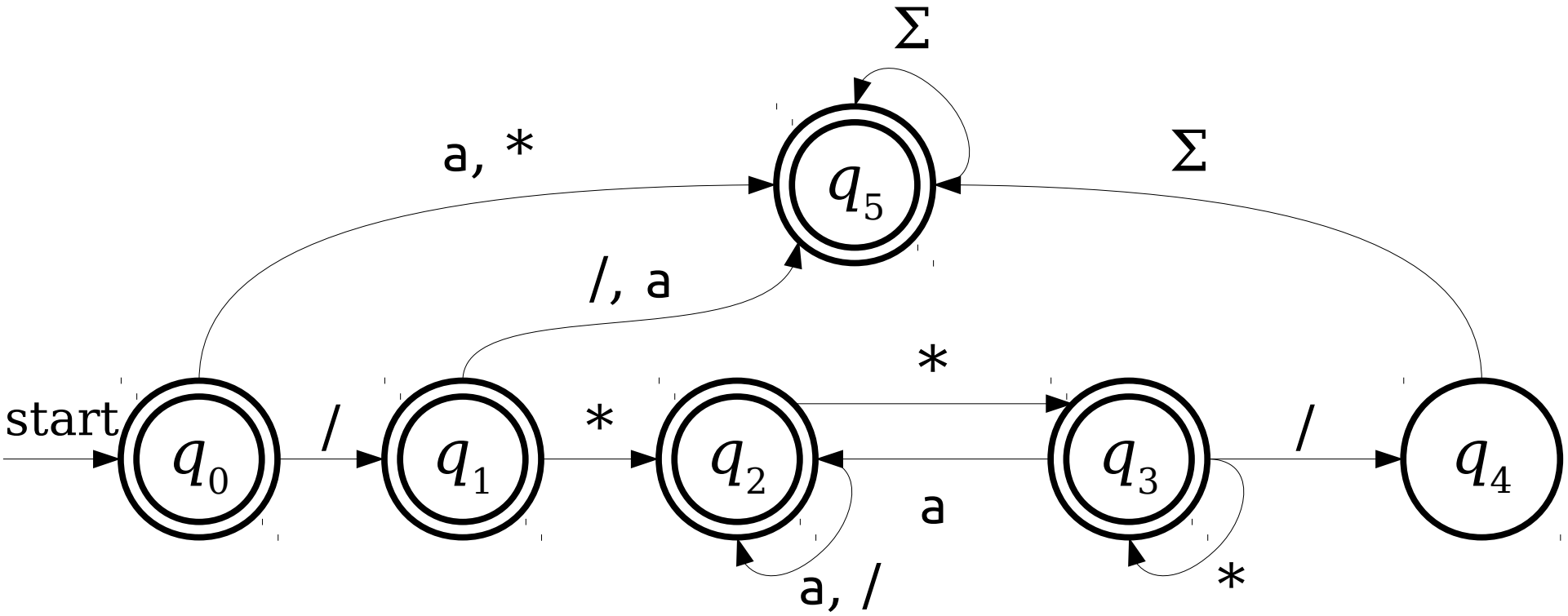
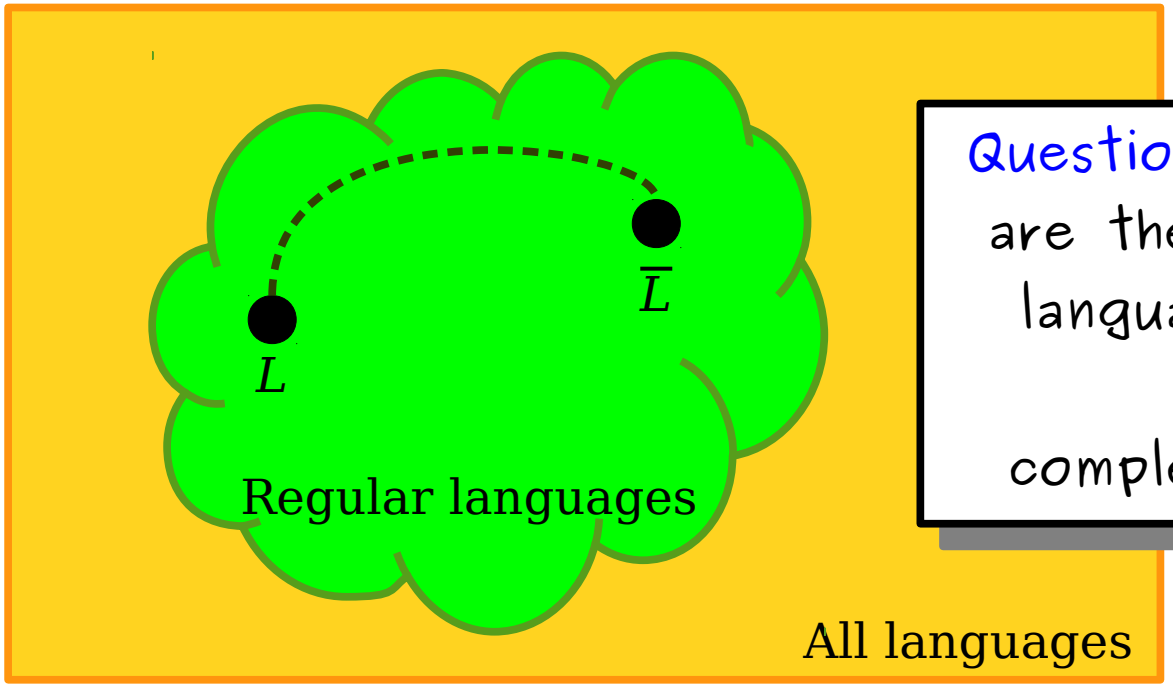


# Closure Properties

- ***Theorem:*** If $L$ is a regular language, then $\overline{L}$ is also a regular language.
- As a result, we say that the regular languages are ***closed under complementation***.




Question to ponder: are the *nonregular* languages closed under complementation?

# The Union of Two Languages

- If $L_1$ and $L_2$ are languages over the alphabet $\Sigma$, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If $L_1$ and $L_2$ are regular languages, is $L_1 \cup L_2$?

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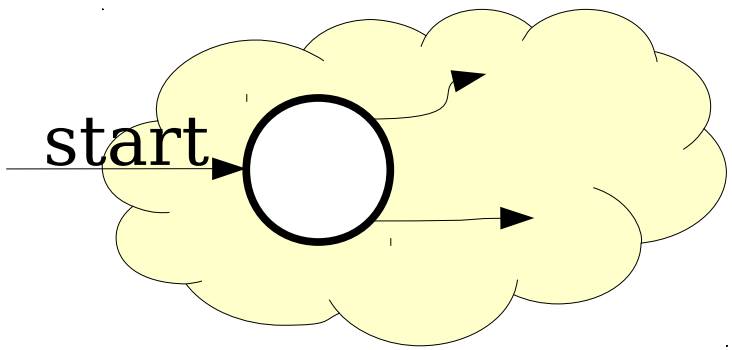
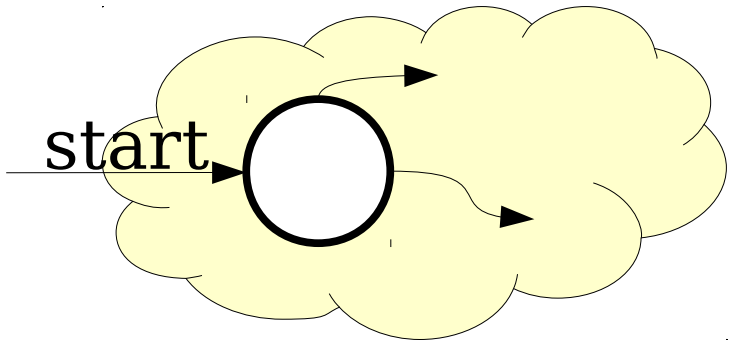



Machine for $L_1$




Machine for $L_2$

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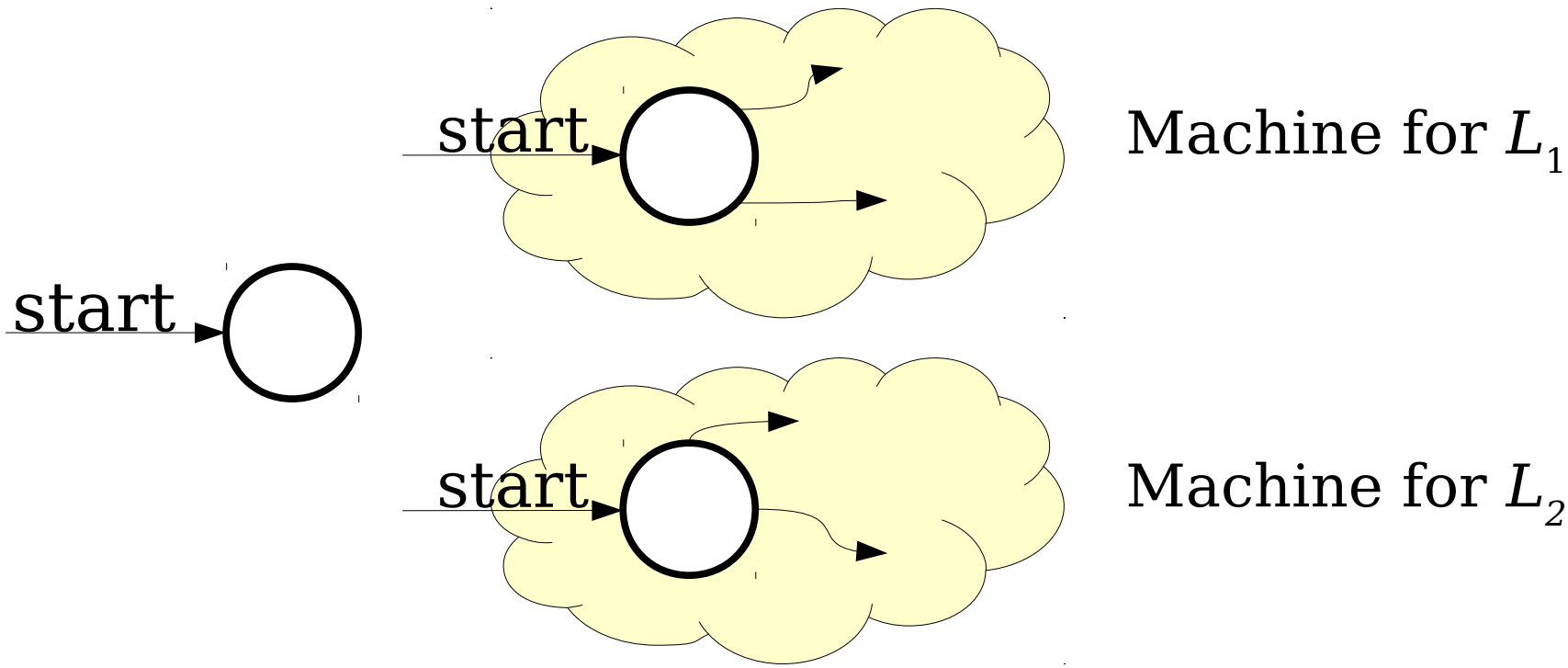

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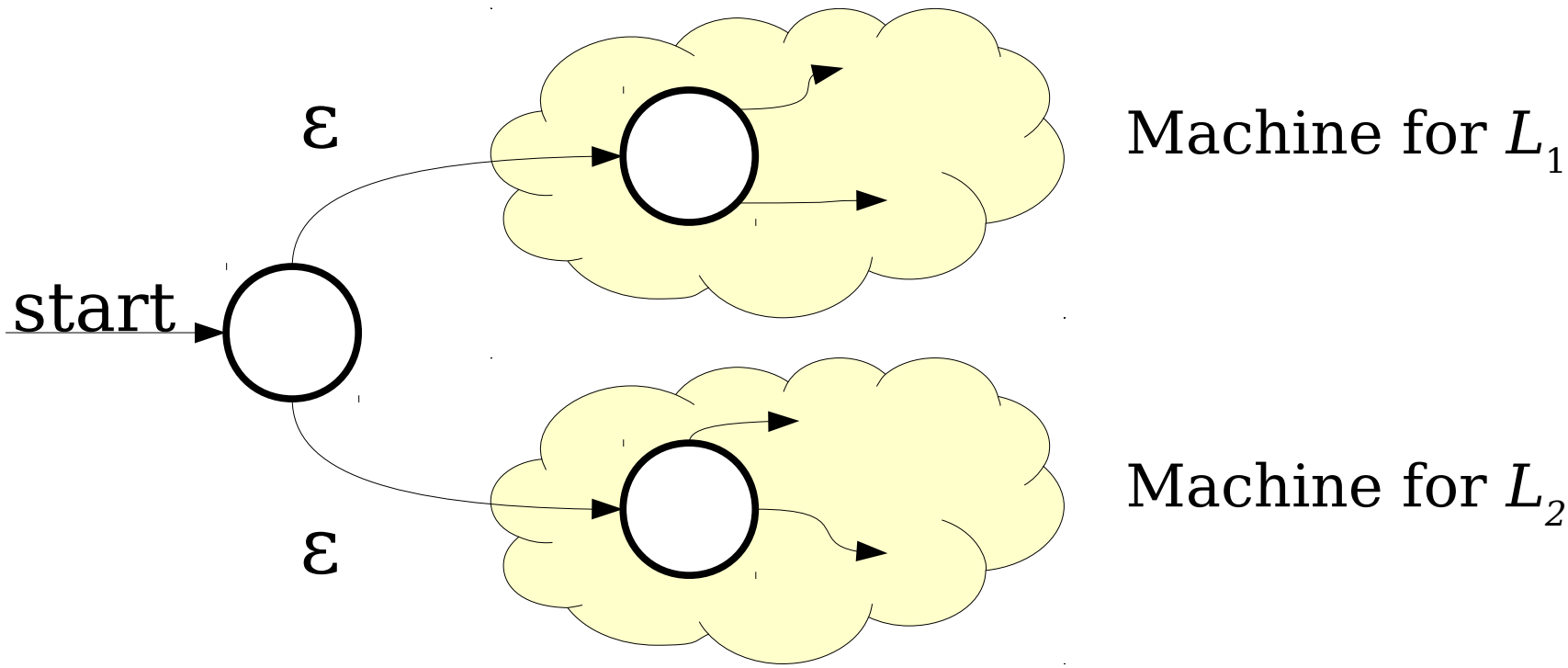

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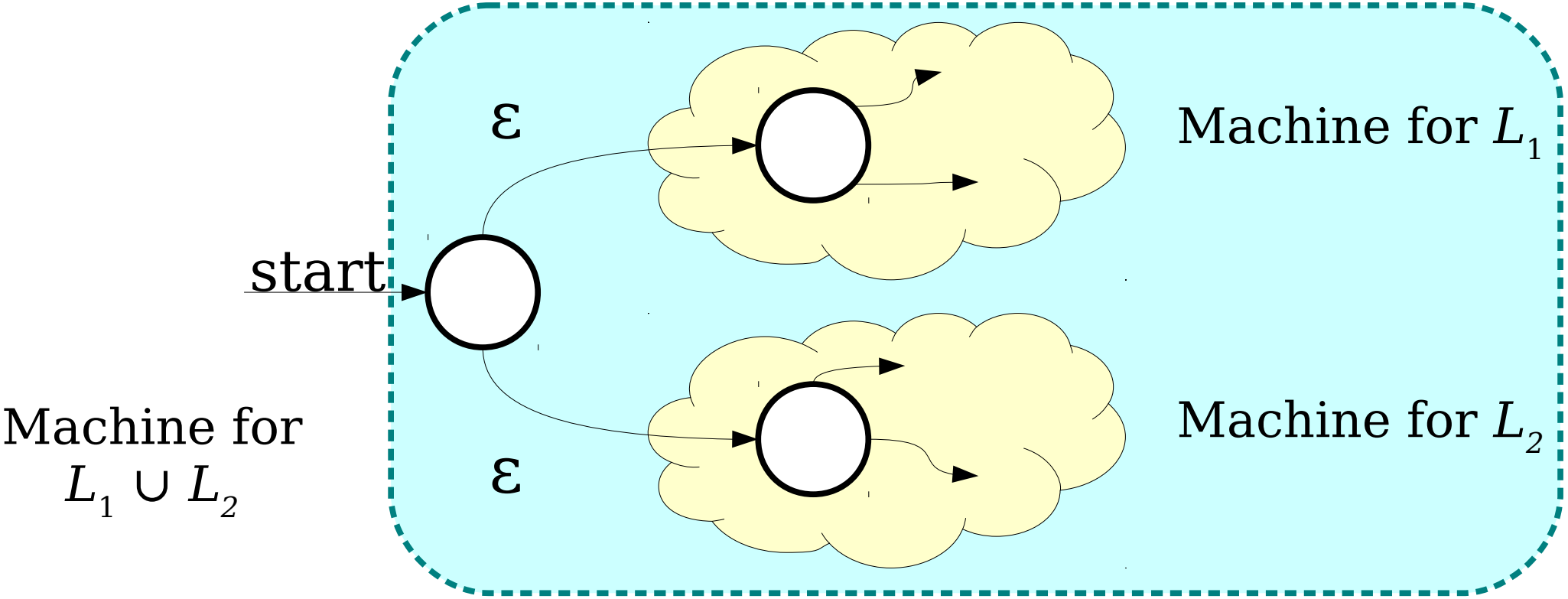

# The Intersection of Two Languages

- If $L_1$ and $L_2$ are languages over $\Sigma$, then $L_1 \cap L_2$ is the language of strings in both $L_1$ and $L_2$.
- Question: If $L_1$ and $L_2$ are regular, is $L_1 \cap L_2$ regular as well?

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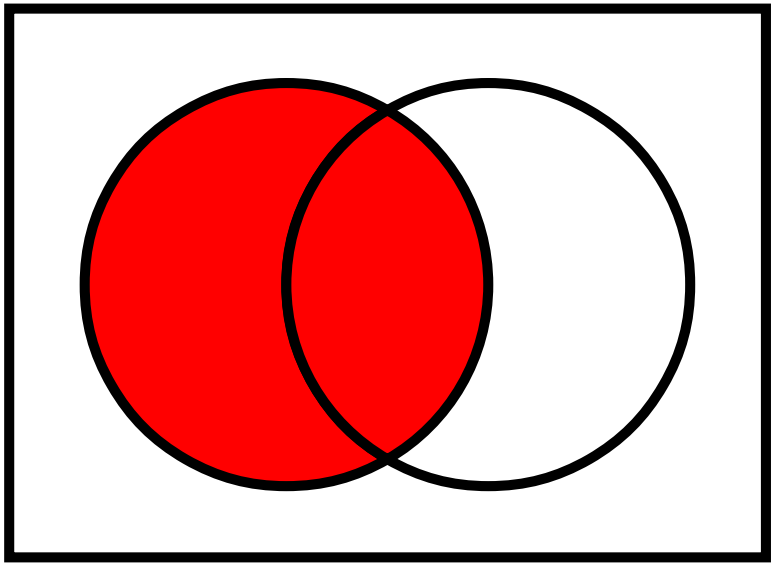
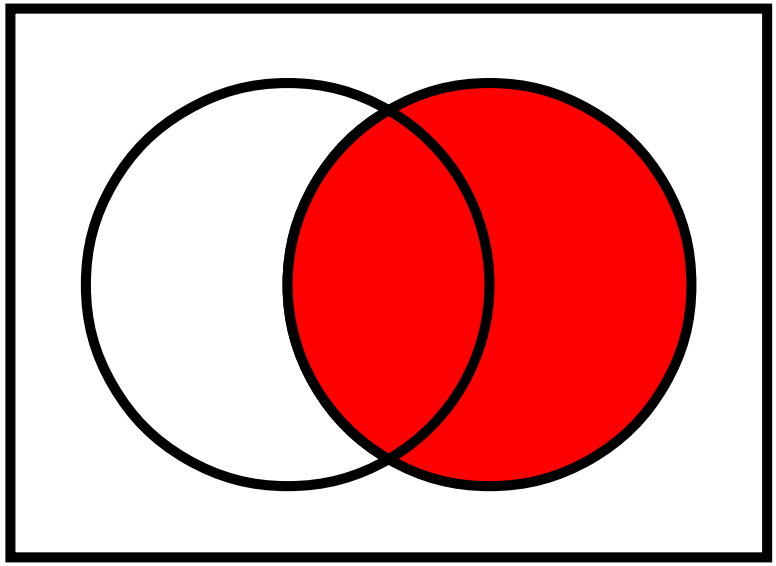


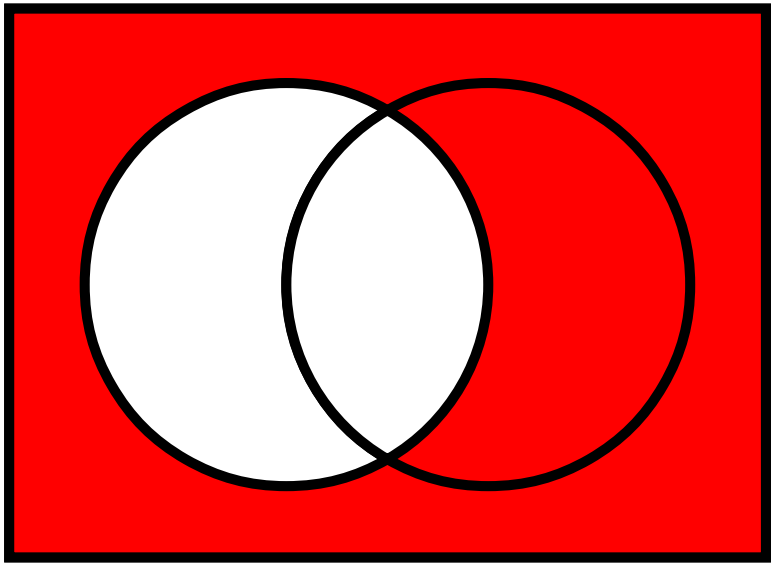
$L_1$



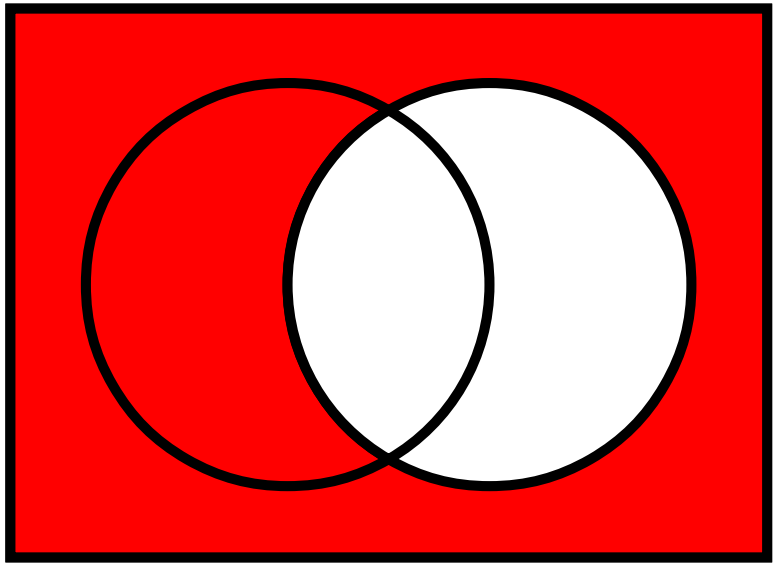
$L_2$

# The Intersection of Two Languages

- If $L_1$ and $L_2$ are languages over $\Sigma$, then $L_1 \cap L_2$ is the language of strings in both $L_1$ and $L_2$.
- Question: If $L_1$ and $L_2$ are regular, is $L_1 \cap L_2$ regular as well?



$\overline{L}_1$



$\overline{L}_2$

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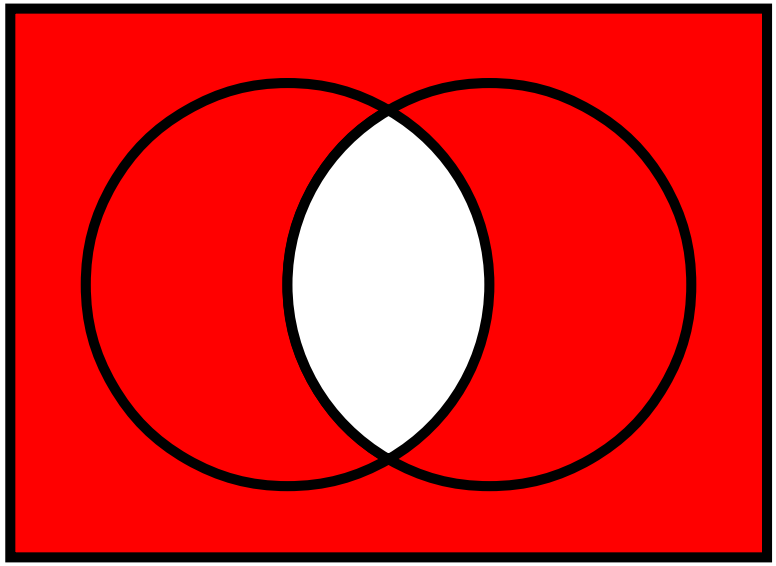

$$\overline{L}_1 \cup \overline{L}_2$$

# The Intersection of Two Languages

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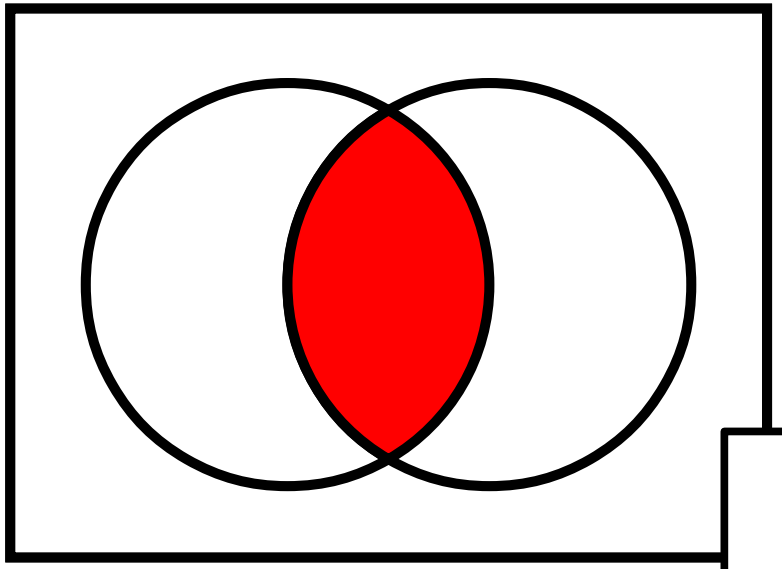

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

# Concatenation

# String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of $w$ and $x$, denoted ***wx***, is the string formed by tacking all the characters of $x$ onto the end of $w$.
- Example: if $w$ = **quo** and $x$ = **kka**, the concatenation $wx$ = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

# Concatenation

- The ***concatenation*** of two languages $L_1$ and $L_2$ over the alphabet $\Sigma$ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

# Concatenation Example

- Let Σ = { `a`, `b`, …, `z`, `A`, `B`, …, `Z` } and consider these languages over Σ:
  - ***Noun*** = { `Puppy`, `Rainbow`, `Whale`, … }
  - ***Verb*** = { `Hugs`, `Juggles`, `Loves`, … }
  - ***The*** = { `The` }
- The language ***TheNounVerbTheNoun*** is
  - { `ThePuppyHugsTheWhale`, `TheWhaleLovesTheRainbow`, `TheRainbowJugglesTheRainbow`, … }

# Concatenation

- The ***concatenation*** of two languages $L_1$ and $L_2$ over the alphabet $\Sigma$ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

- Two views of $L_1L_2$:
  - The set of all strings that can be made by concatenating a string in $L_1$ with a string in $L_2$.
  - The set of strings that can be split into two pieces: a piece from $L_1$ and a piece from $L_2$.

# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
- Intuition – can we split a string $w$ into two strings $xy$ such that $x \in L_1$ and $y \in L_2$?

# Concatenating Regular Languages

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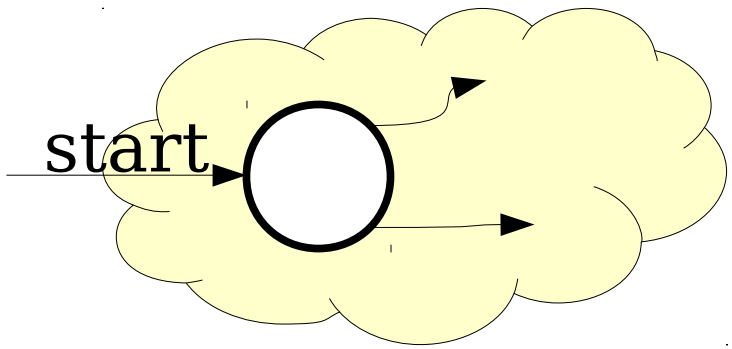
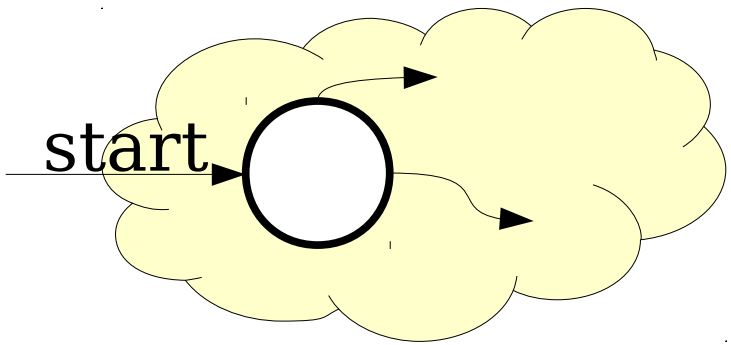



Machine for $L_1$




Machine for $L_2$

# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
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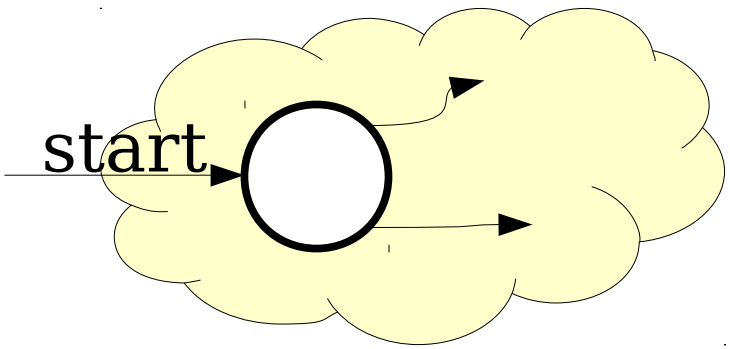


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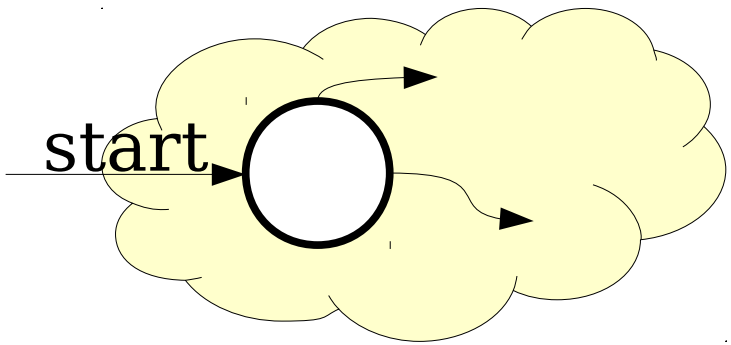


Machine for $L_2$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **b** | **o** | **o** | **k** | **k** | **e** | **e** | **p** | **e** | **r** |

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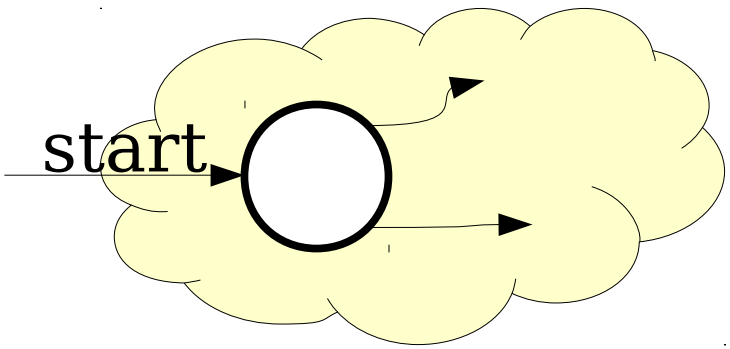


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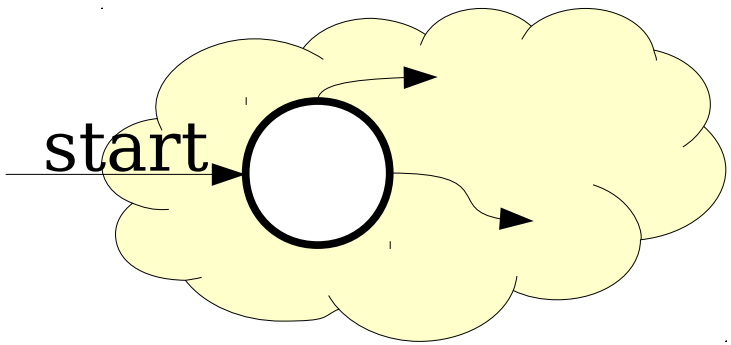


Machine for $L_2$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
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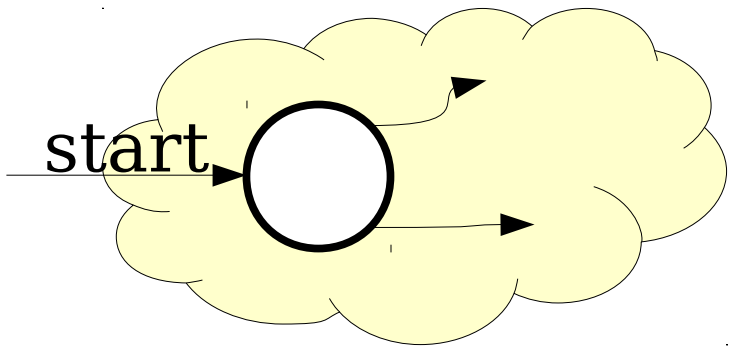


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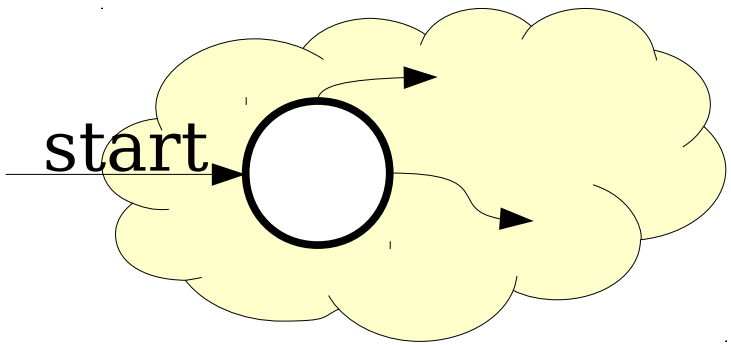



Machine for $L_2$

| | | | |
|---|---|---|---|
| **b** | **o** | **o** | **k** |

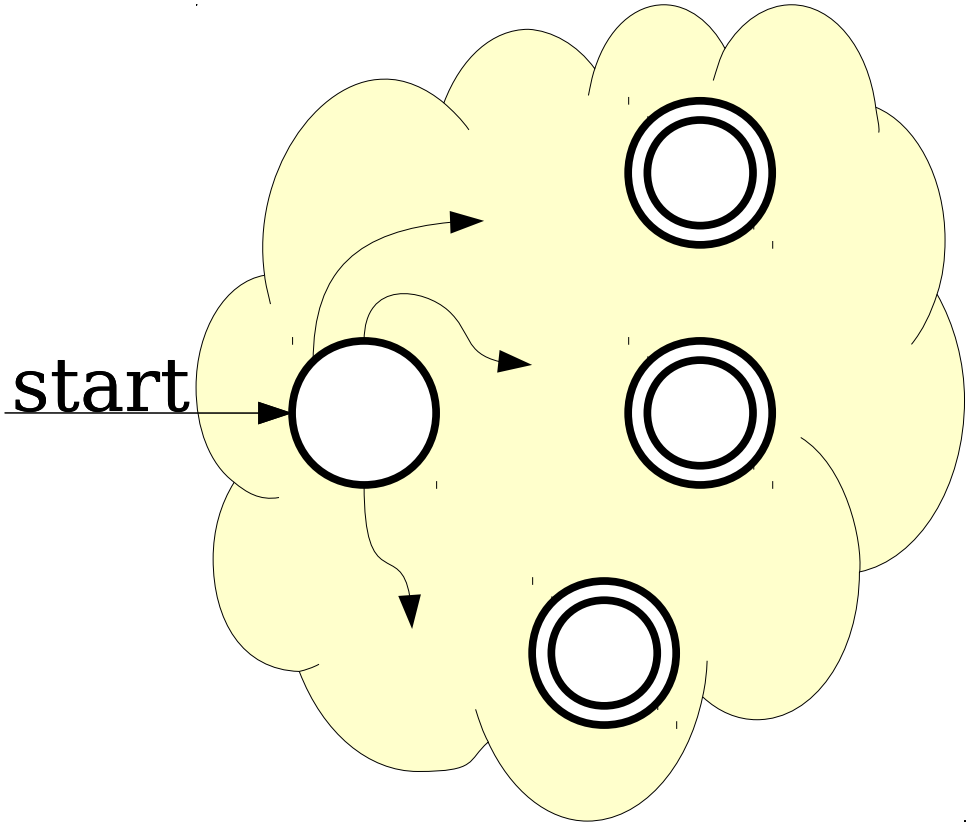
| | | | | | |
|---|---|---|---|---|---|
| **k** | **e** | **e** | **p** | **e** | **r** |

# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
- Intuition – can we split a string $w$ into two strings $xy$ such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for $L_1$ on $w$.
  - Whenever it reaches an accepting state, optionally hand the rest of $w$ to a DFA/NFA for $L_2$.
  - If the automaton for $L_2$ accepts the rest, $w \in L_1L_2$.
  - If the automaton for $L_2$ rejects the remainder, the split was incorrect.

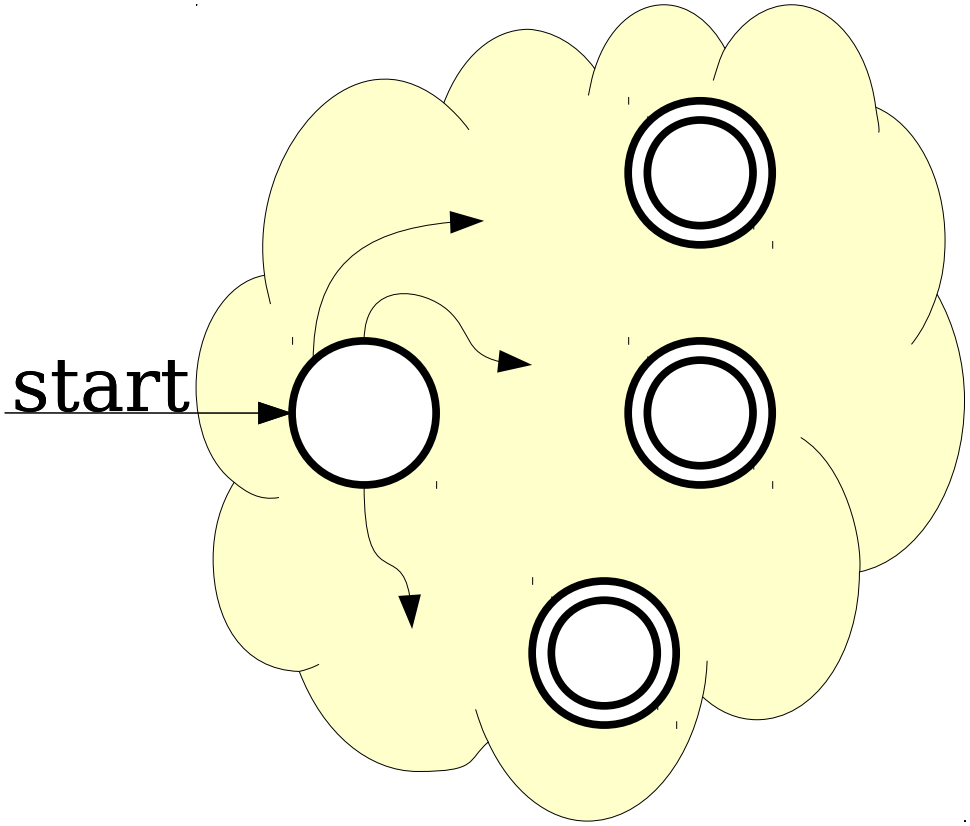
# Concatenating Regular Languages

# Concatenating Regular Languages

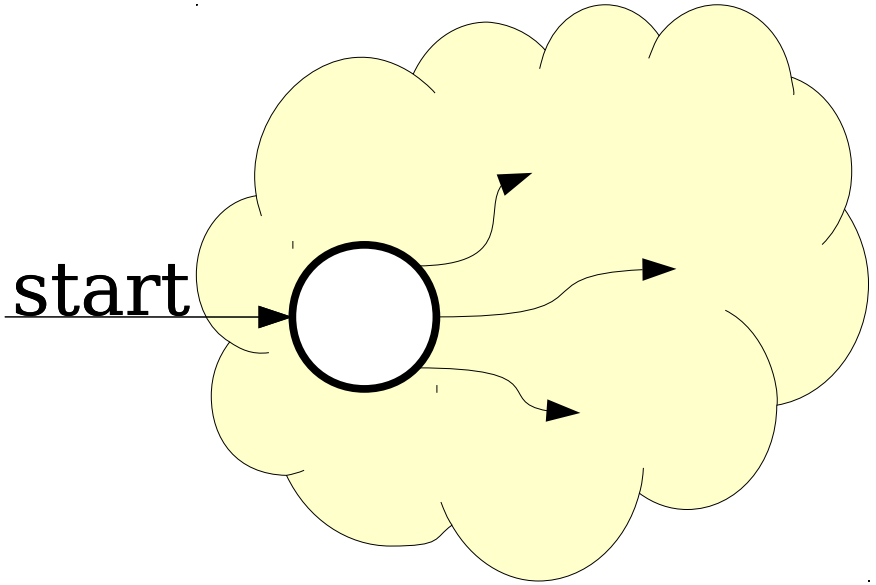



Machine for $L_1$

# Concatenating Regular Languages

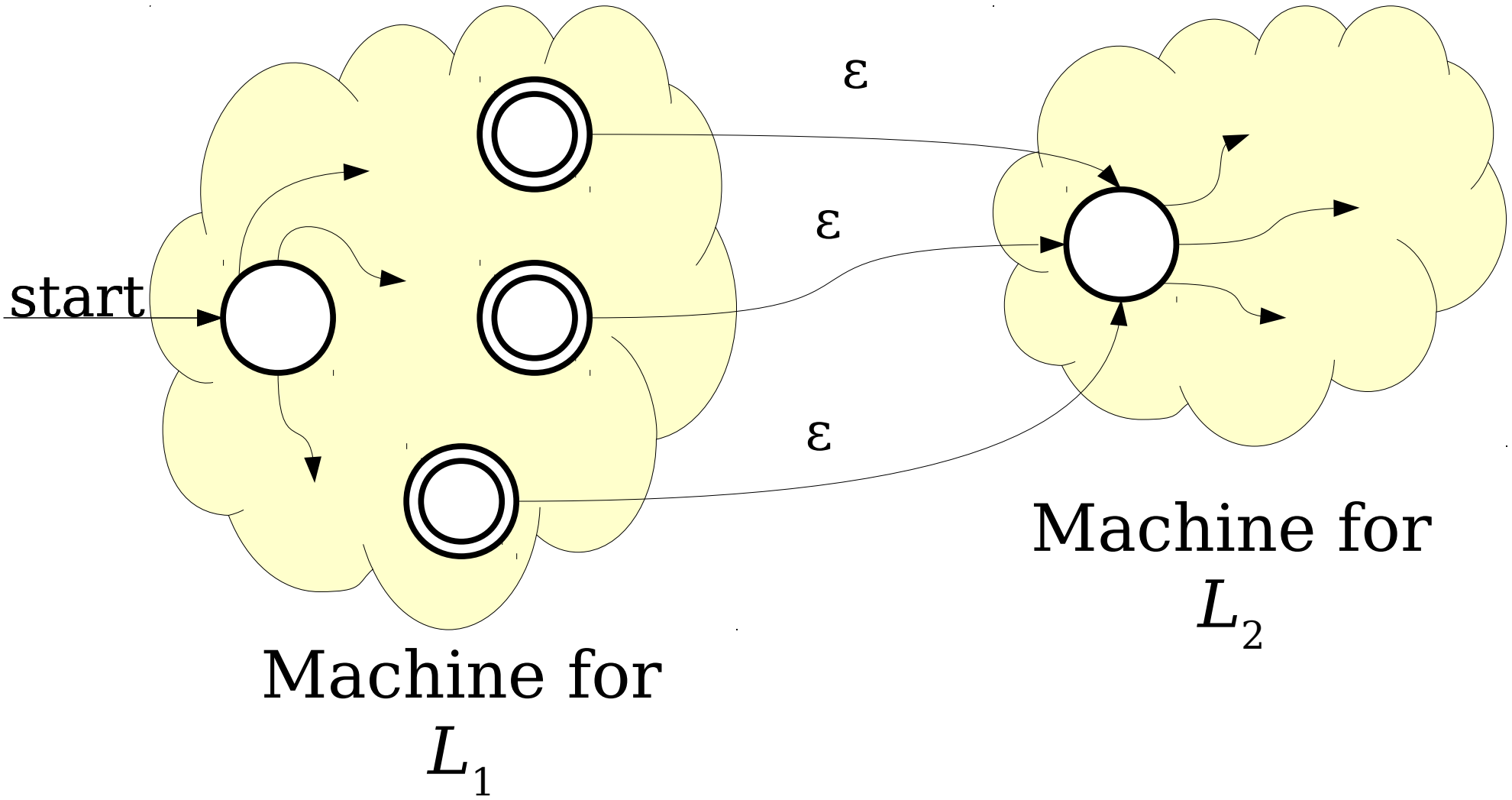



Machine for $L_1$




Machine for $L_2$

# Concatenating Regular Languages



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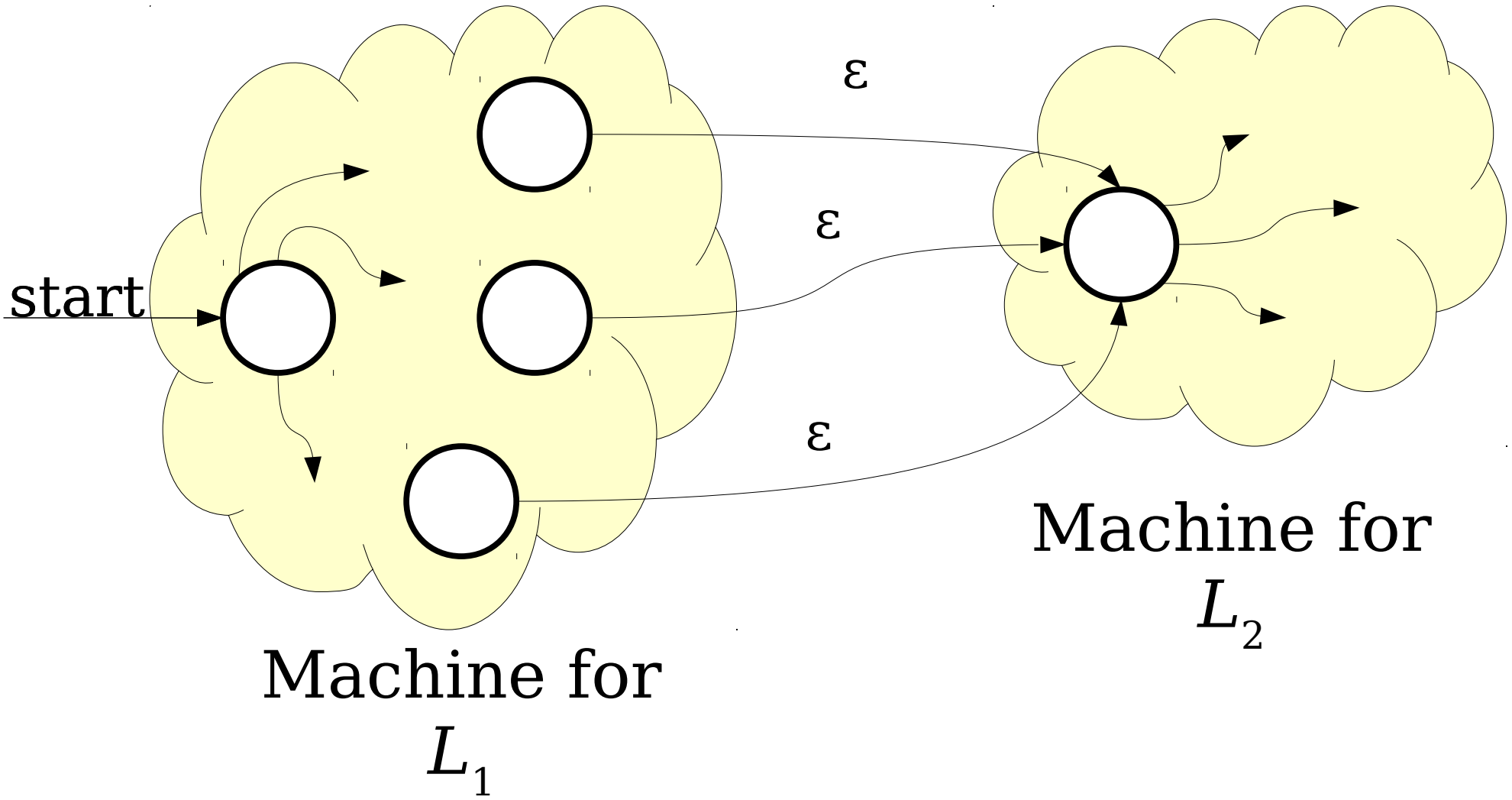

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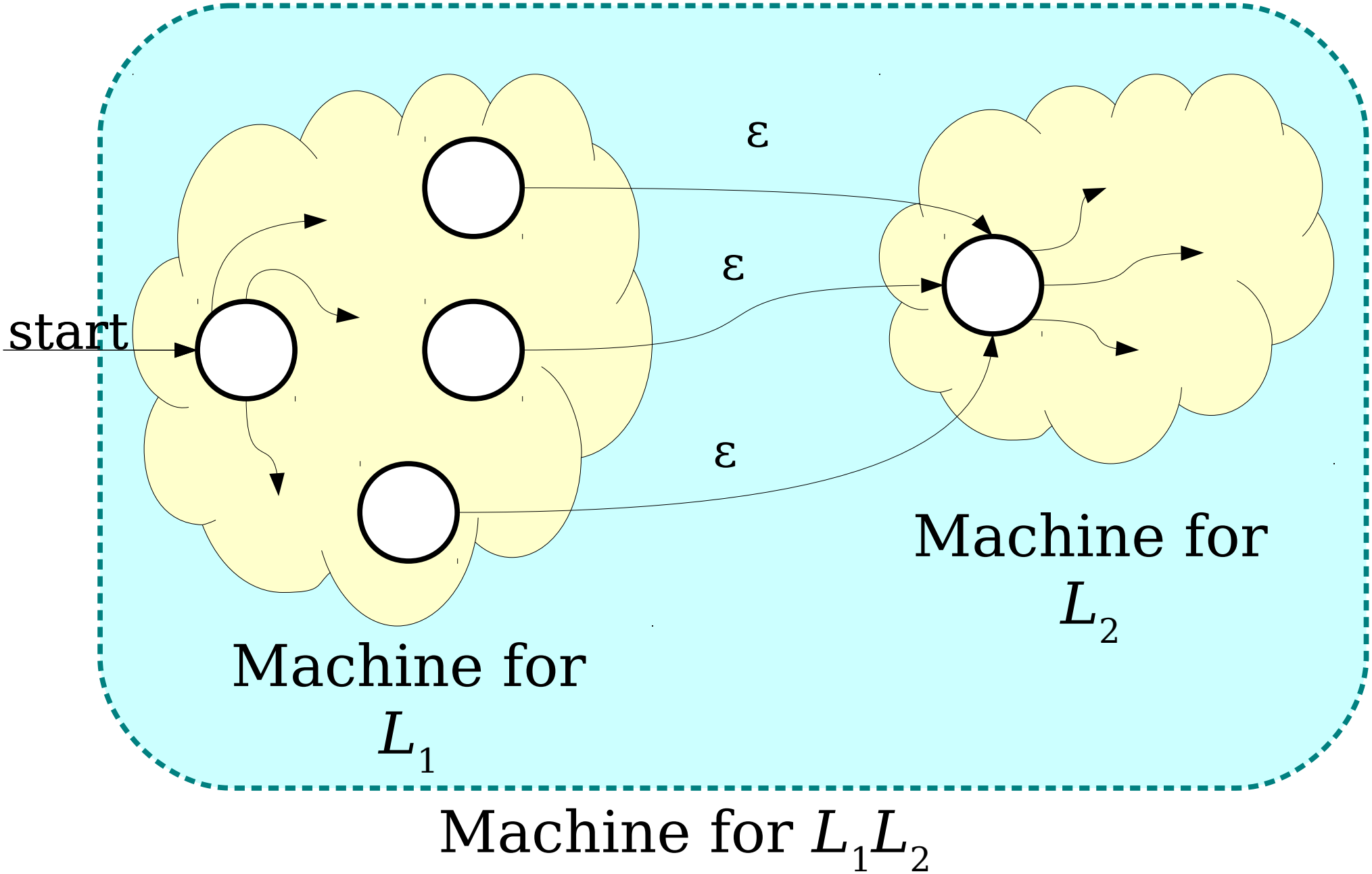


# Lots and Lots of Concatenation

- Consider the language $L$ = { **aa**, **b** }
- $LL$ is the set of strings formed by concatenating pairs of strings in $L$.

  { **aaaa**, **aab**, **baa**, **bb** }

- $LLL$ is the set of strings formed by concatenating triples of strings in $L$.

  { **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in $L$.

  { **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

# Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \emptyset$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating $(n+1)$ strings together works by concatenating $n$ strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\emptyset^0$?

# The Kleene Star

# The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, $L^*$ is the language all possible ways of concatenating zero or more strings in $L$ together, possibly with repetition.
- ***Question to ponder:*** What is $\emptyset^*$?

# The Kleene Closure

If $L$ = { **a**, **bb** }, then $L^*$ = {

ε,

**a**, **bb**,

**aa**, **abb**, **bba**, **bbbb**,

**aaa**, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

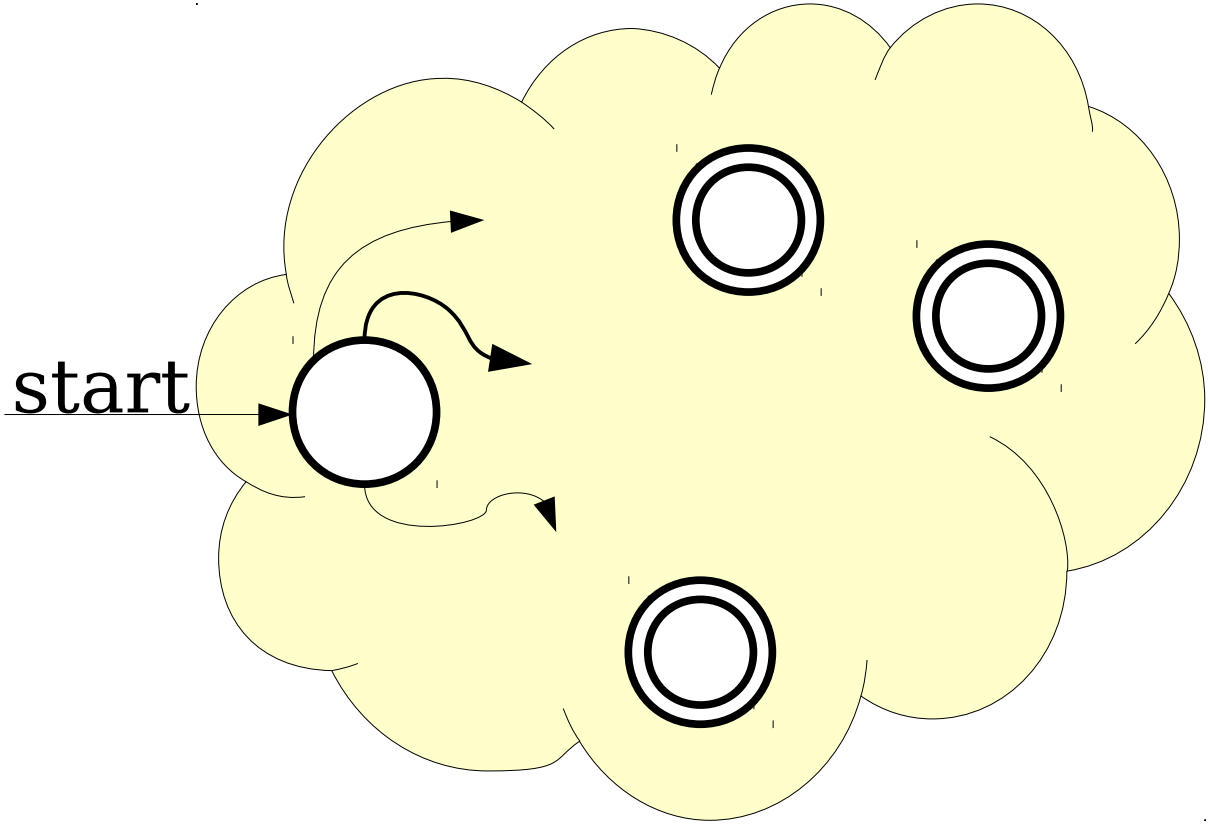
...

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

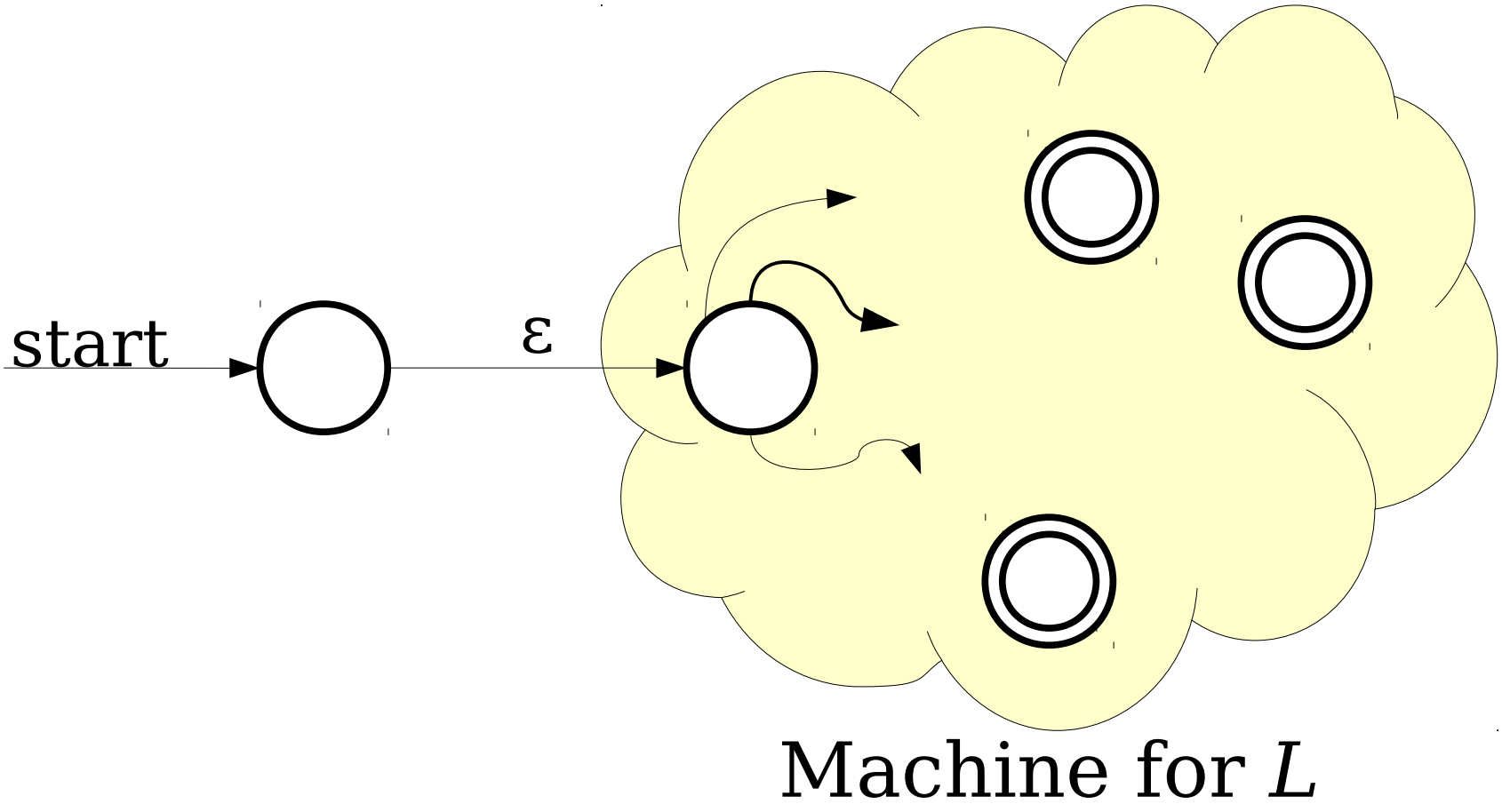
***Idea:*** Can we convert an NFA for language $L$ to an NFA for language $L^*$?

# The Kleene Star




Machine for $L$

# The Kleene Star



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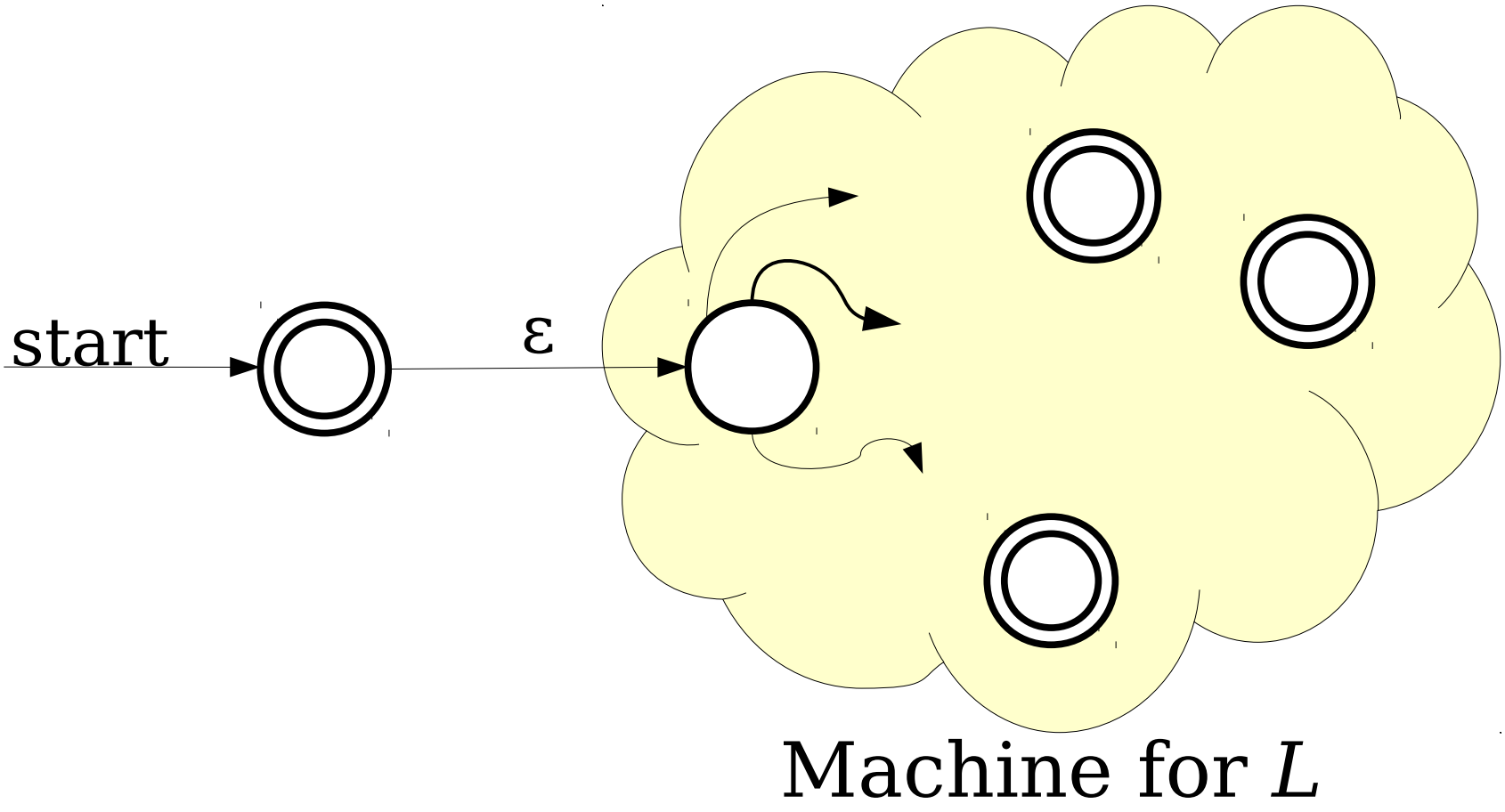

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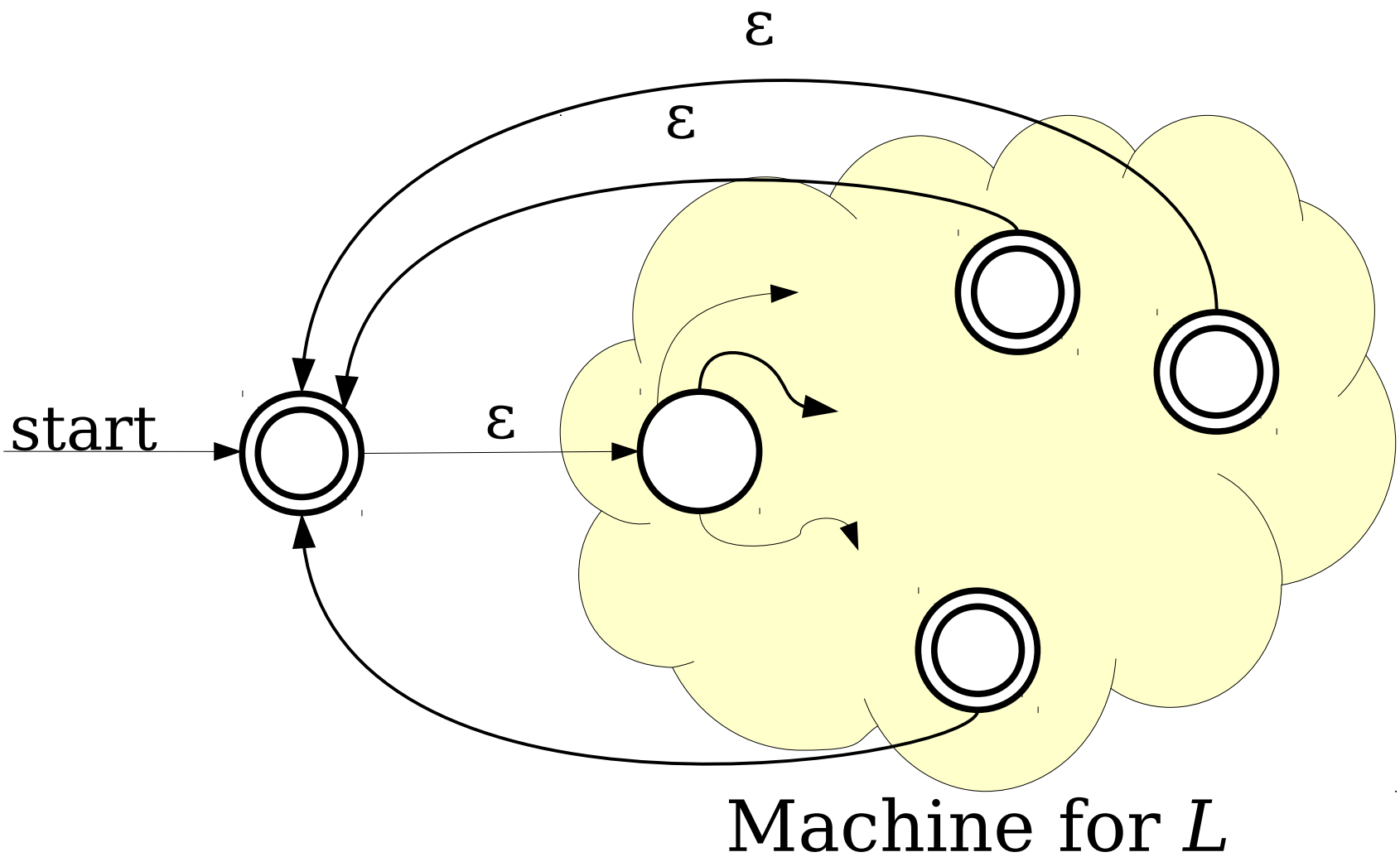

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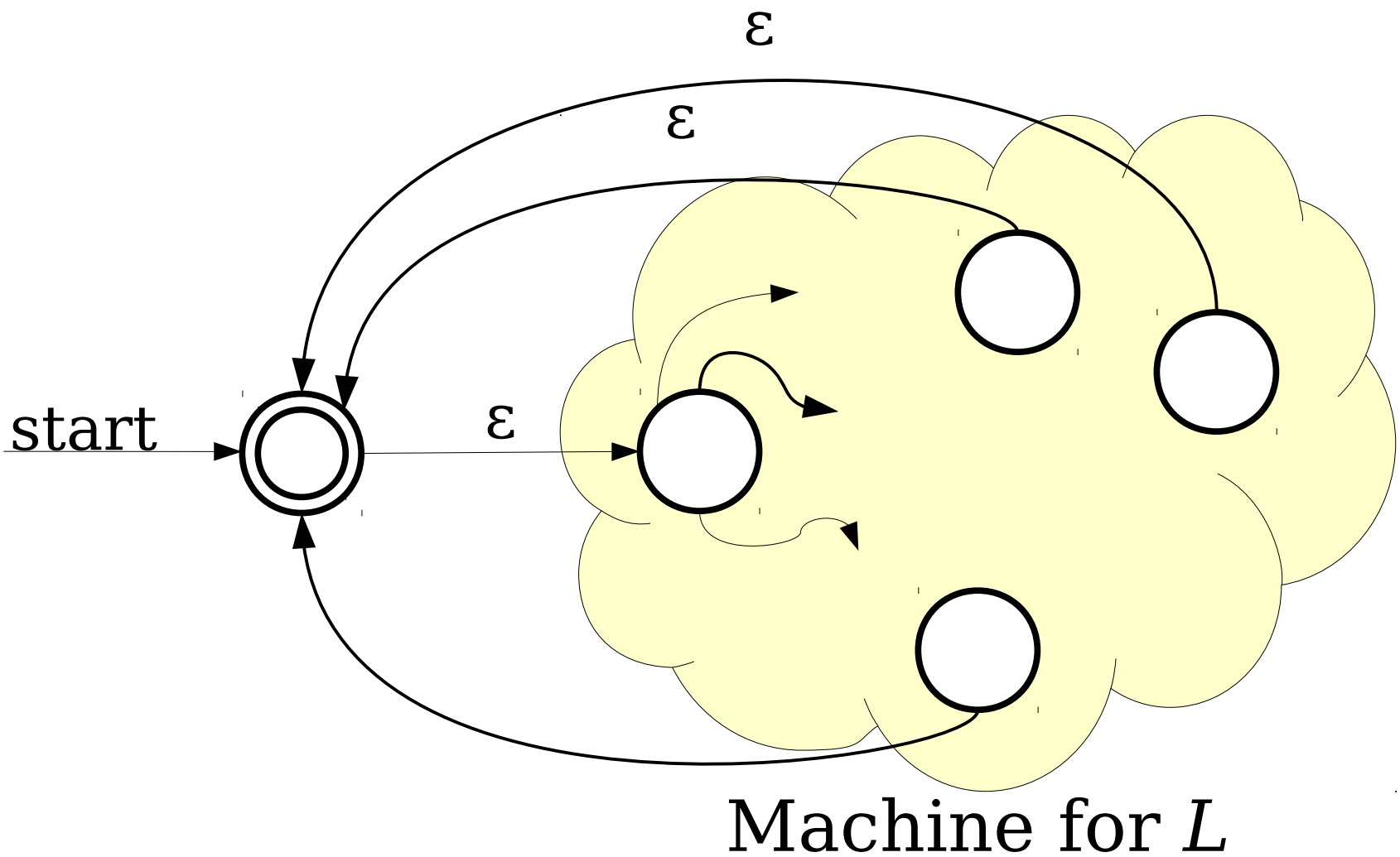

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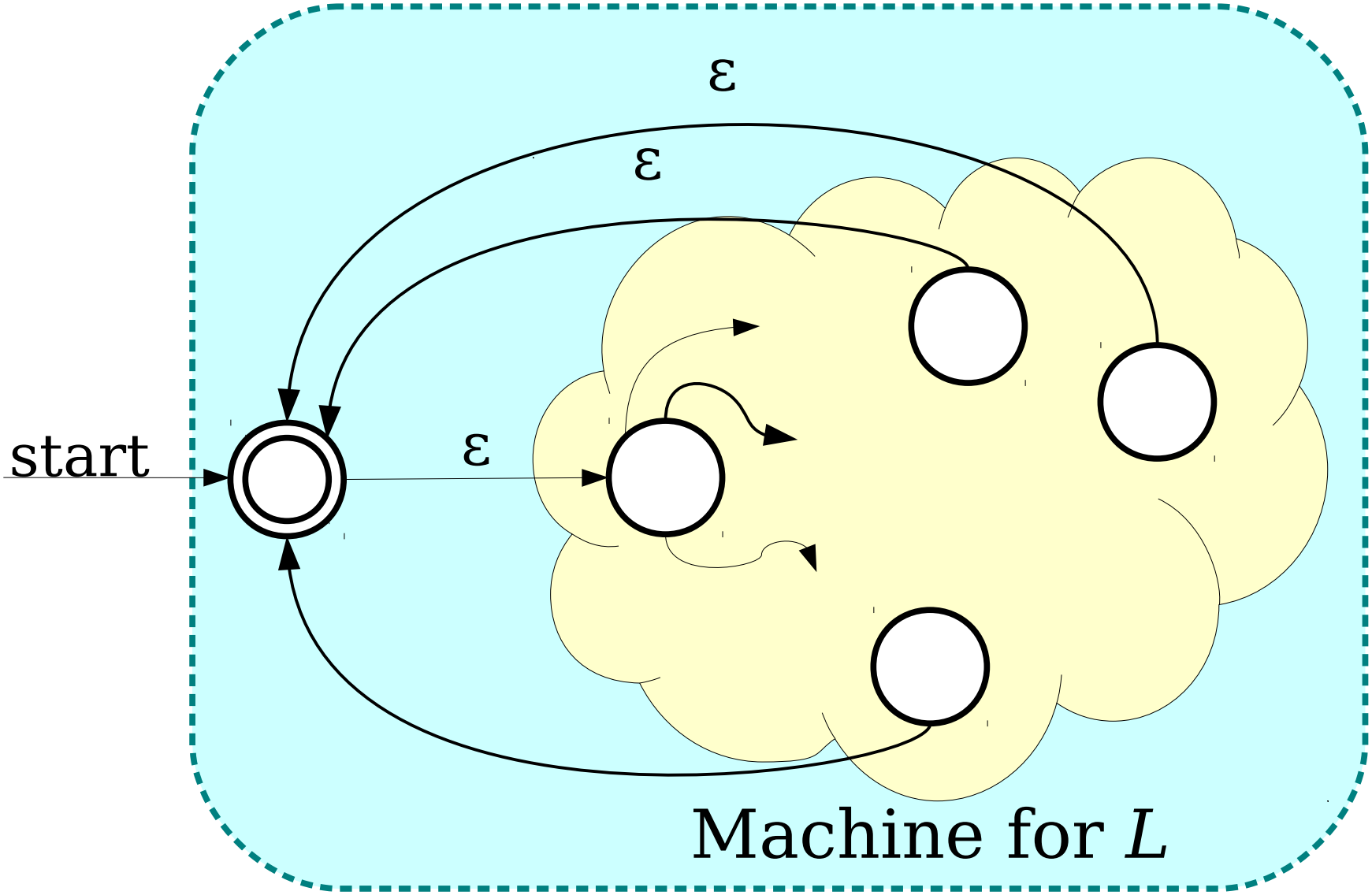
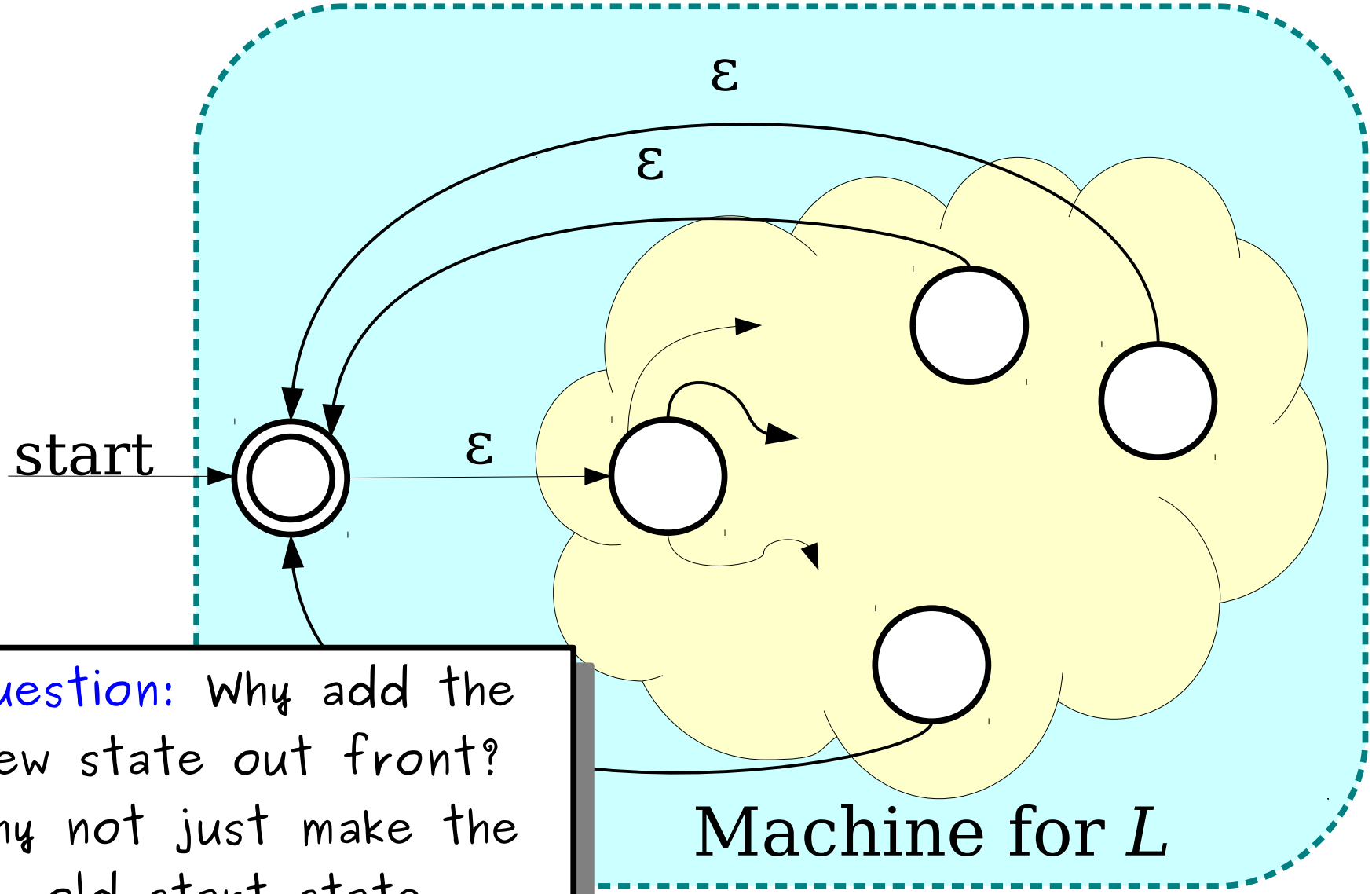



Machine for $L^*$

# The Kleene Star




Question: Why add the new state out front? Why not just make the old start state accepting?

achine for $L^*$

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet $\Sigma$, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

# Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!